

Padé Approximants, density of rational functions in $A^\infty(\Omega)$ and smoothness of the integration operator

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Abstract

First we establish some generic universalities for Padé approximants in the closure $X^\infty(\Omega)$ in $A^\infty(\Omega)$ of all rational functions with poles off $\overline{\Omega}$, the closure taken in \mathbb{C} of the domain $\Omega \subset \mathbb{C}$. Next we give sufficient conditions on Ω so that $X^\infty(\Omega) = A^\infty(\Omega)$. Some of these conditions imply that, even if the boundary $\partial\Omega$ of a Jordan domain Ω has infinite length, the integration operator on Ω preserves $H^\infty(\Omega)$ and $A(\Omega)$ as well. We also give an example of a Jordan domain Ω and a function $f \in A(\Omega)$, such that its antiderivative is not bounded on Ω . Finally we restate these results for Volterra operators on the open unit disc D and we complete them by some generic results.

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1 Introduction

Universality of Taylor series is a generic phenomenon, ([14], [10], [8], [4], [13], [11]), where the partial sums of the Taylor expansion of a holomorphic function f approximate many functions on compact sets outside the domain of definition Ω of f . Recently the partial sums of the Taylor expansion of f , which are polynomials, have been replaced by some Padé approximants of the Taylor series of f , which are rational functions and may take the value ∞ as well ([15]). This time the approximation is uniform on compact sets

K with respect to the chordal metric χ on $\mathbb{C} \cup \{\infty\}$. If the compact sets K are disjoint from $\overline{\Omega}$, then the universal function f can be chosen to be smooth on the boundary of Ω , that is, $f \in A^\infty(\Omega)$. In addition f can be approximated by rational functions with poles off $\overline{\Omega}$ in the natural topology of $A^\infty(\Omega)$; that is, $f \in X^\infty(\Omega)$, where $X^\infty(\Omega)$ denotes the closure in $A^\infty(\Omega)$ of the set of rational functions with poles off $\overline{\Omega}$. Thus, we obtain generic universalities of Padé approximants in $X^\infty(\Omega)$; this is established in Sections 3 and 4. However, generic results on closed subspaces of $A^\infty(\Omega)$ may be considered as simple existence results and therefore, they are less significant than generic results on the whole space $A^\infty(\Omega)$. That is why in Section 5 we give sufficient conditions such that $X^\infty(\Omega) = A^\infty(\Omega)$. A simple form of these conditions is that $(\overline{\Omega})^0 = \Omega$, $\mathbb{C} \setminus \overline{\Omega}$ is connected and that there exists a constant $M < +\infty$ such that any two points of Ω can be joined in Ω by a curve with length at most M . Under these hypotheses polynomials are dense in $A^\infty(\Omega)$. If Ω is a Jordan domain with rectifiable boundary then the above condition is fulfilled [13]; however, it is possible that a Jordan domain Ω , whose boundary has infinite length, satisfies the previous sufficient condition and therefore $X^\infty(\Omega) = A^\infty(\Omega)$. Such examples are all starlike Jordan domains Ω where $\partial\Omega$ has infinite length.

If Ω is any Jordan domain Ω , where $\partial\Omega$ has finite length, then it is known that the integration operator on Ω is smooth. That is, if $f \in H^\infty(\Omega)$ is a bounded holomorphic function, then its antiderivative F ($F' = f$ on Ω) belongs to $A(\Omega)$ and extends continuously on $\overline{\Omega}$. In Section 6 we investigate the smoothness of the integration operator $H(\Omega) \ni f \rightarrow F(f) \in H(\Omega)$ where $F'(f) = f$ on Ω and $F(f)(z_0) = 0$ for some fixed point $z_0 \in \Omega$. We give sufficient conditions of the previous type so that $F(f) \in H^\infty(\Omega)$ for all $f \in H^\infty(\Omega)$, as well as $F(f) \in A(\Omega)$ for all $f \in A(\Omega)$. This may occur even if $\partial\Omega$ has infinite length. Furthermore, we give a specific example of a Jordan domain Ω and a function $f \in A(\Omega)$ so that $F(f) \notin H^\infty(\Omega)$. This relates to the standard singular inner function $\exp \frac{z+1}{z-1}$, which has previously been used by one of the authors (see [16] and [[17] Prop. 19]). In the above example the boundary of Ω contains only one “bad” point, which can not be reached from an interior point using a curve in Ω with finite length. Also the constructed function f is almost explicit. At the end of Section 6 we reformulate the previous results in the language of Volterra operators on the open unit disc D (see [2] and the references therein).

In Section 7 we give generic versions of the results of Section 6. For instance for any

Jordan domain Ω we show that the set of functions $f \in A(\Omega)$ such that $F(f) \notin H^\infty(\Omega)$ is either empty or large in the topological sense, that is G_δ and dense in $A(\Omega)$ endowed with the topology of supremum norm on Ω . We also obtain a result in this direction for Volterra operators on the open unit disc D . Finally we show that for all holomorphic functions g in a dense subset of $H(D)$ (respectively $A(D)$), there exists $f \in A(D)$ such that $T_g(f) \notin H^\infty(D)$, where $T_g(f)$ is the antiderivative on D of fg' vanishing at 0. An open question is to find a complete metric topology in the set of all Jordan domains (contained in \overline{D}), so that for the generic Jordan domain Ω , there exists $f \in A(\Omega)$ whose antiderivative F is not bounded in Ω (or at least $F \notin A(\Omega)$).

2 Preliminaries

Let $\Omega \subset \mathbb{C}$ be open. We say that a holomorphic function f defined on Ω , belongs to $A^\infty(\Omega)$ if and only if for every $\ell \in \{0, 1, 2, \dots\}$ the ℓ th derivative $f^{(\ell)}$ extends continuously on $\overline{\Omega}$.

In $A^\infty(\Omega)$, we consider the topology defined by the seminorms $\sup_{z \in K_n} |f^{(\ell)}(z)|$, where $\ell \in \{0, 1, 2, \dots\}$, and $(K_n)_{n \in \mathbb{N}}$ is a family of compact sets in $\overline{\Omega}$, such that for every compact set L in $\overline{\Omega}$ there exists $n \in \mathbb{N}$ with $L \subset K_n$. Such a family is for example the family of the sets $\overline{\Omega} \cap \overline{D(0, n)}$, $n \in \mathbb{N}$. With this topology $A^\infty(\Omega)$ becomes a Fréchet space.

Now we call $X^\infty(\Omega)$, the closure in $A^\infty(\Omega)$ of all rational functions with poles off $\overline{\Omega}$, where the closure is taken in \mathbb{C} .

If we consider the one point compactification $\mathbb{C} \cup \{\infty\} = \tilde{\mathbb{C}}$ of \mathbb{C} , then a well known metric is the chordal metric χ on $\mathbb{C} \cup \{\infty\}$, where

$$\chi(a, b) = \frac{|a - b|}{\sqrt{1 + |a|^2} \sqrt{1 + |b|^2}}, \quad \text{for } a, b \in \mathbb{C}$$

and $\chi(a, \infty) = \frac{1}{\sqrt{1 + |a|^2}}$ for $a \in \mathbb{C}$, and $\chi(\infty, \infty) = 0$; see [1].

Proposition 2.1. *Let $K \subset \mathbb{C}$ be a compact set and $q = \frac{A}{B}$ a rational function, where the polynomials A, B do not have a common root in \mathbb{C} . Then there is a sequence $q_j = \frac{A_j}{B_j}$, $j = 1, 2, \dots$ where the polynomials A_j and B_j have coefficients in $\mathbb{Q} + i\mathbb{Q}$ and do not have any common zero in \mathbb{C} for all j , such that $\sup_{z \in K} \chi(q_j(z), q(z)) \rightarrow 0$ as $j \rightarrow +\infty$.*

The above proposition is well known. See [15].

Let $\zeta \in \mathbb{C}$ be fixed and

$$f = \sum_{n=0}^{\infty} a_n (z - \zeta)^n$$

be a formal power series ($a_n = a_n(f, \zeta)$). Often this power series is the Taylor development of a holomorphic function f in a neighborhood of ζ . Let p and q be two non negative integers. The Padé approximant $[f; p/q]_{\zeta}(z)$ is defined to be a rational function ϕ regular at ζ whose Taylor development with center ζ ,

$$\phi(z) = \sum_{n=0}^{\infty} b_n (z - \zeta)^n,$$

satisfies $b_n = a_n$ for all $0 \leq n \leq p+q$ and $\phi(z) = A(z)/B(z)$, where the polynomials A and B satisfy

$$\deg A \leq p, \quad \deg B \leq q \quad \text{and} \quad B(\zeta) \neq 0.$$

It is not always true that such a rational function ϕ exists. And if it exists it is not always unique. For $q = 0$, we always have such a unique ϕ which is

$$[f; p/q]_{\zeta}(z) = \sum_{n=0}^p a_n (z - \zeta)^n.$$

For $q \geq 1$ the necessary and sufficient condition for existence and uniqueness is that the following $q \times q$ Hankel determinant is non-zero ([3])

$$\begin{vmatrix} a_{p-q+1} & a_{p-q+2} & \cdot & \cdot & a_p \\ a_{p-q+2} & a_{p-q+3} & \cdot & \cdot & a_{p+1} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_p & a_{p+1} & \cdot & \cdot & a_{p+q-1} \end{vmatrix} \neq 0,$$

where $a_i = 0$ for $i < 0$. If this is satisfied we write $f \in \mathcal{D}_{p,q}(\zeta)$. For $f \in \mathcal{D}_{p,q}(\zeta)$ the Padé approximant

$$[f; p/q]_{\zeta}(z) = \frac{A(f, \zeta)(z)}{B(f, \zeta)(z)}$$

is given by the following Jacobi formula

$$A(f, \zeta)(z) = \begin{vmatrix} (z - \zeta)^q S_{p-q}(f, \zeta)(z) & (z - \zeta)^{q-1} S_{p-q+1}(f, \zeta)(z) & \cdot & \cdot & S_p(f, \zeta)(z) \\ a_{p-q+1} & a_{p-q+1} & \cdot & \cdot & a_{p+1} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_p & a_{p+1} & \cdot & \cdot & a_{p+q} \end{vmatrix},$$

$$B(f, \zeta)(z) = \begin{vmatrix} (z - \zeta)^q & (z - \zeta)^{q-1} & \cdot & \cdot & 1 \\ a_{p-q+1} & a_{p-q+1} & \cdot & \cdot & a_{p+1} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_p & a_{p+1} & \cdot & \cdot & a_{p+q} \end{vmatrix},$$

with (see [3])

$$S_k(f, \zeta)(z) = \begin{cases} \sum_{\nu=0}^k a_\nu (z - \zeta)^\nu, & \text{if } k \geq 0 \\ 0, & \text{if } k < 0. \end{cases}$$

If $A(f, \zeta)(z)$ and $B(f, \zeta)(z)$ are given by the previous Jacobi formula and they do not have a common zero in a set K we write $f \in E_{p,q,\zeta}(K)$. Equivalently

$$|A(f, \zeta)(z)|^2 + |B(f, \zeta)(z)|^2 \neq 0$$

for all $z \in K$. For K compact this is equivalent to the existence of a $\delta > 0$ such that

$$|A(f, \zeta)(z)|^2 + |B(f, \zeta)(z)|^2 > \delta$$

for all $z \in K$. We will also use the following ([3] Th. 1.4.4 page 30).

Proposition 2.2. *Let $\phi(z) = \frac{A(z)}{B(z)}$ be a rational function, where the polynomials A and B do not have any common zero in \mathbb{C} . Let $\deg A(z) = k$ and $\deg B(z) = \lambda$. Then for every $\zeta \in \mathbb{C}$ such that $B(\zeta) \neq 0$ we have the following:*

$$\phi \in \mathcal{D}_{k,\lambda}(\zeta),$$

$$\phi \in \mathcal{D}_{p,\lambda}(\zeta) \text{ for all } p > k,$$

$$\phi \in \mathcal{D}_{k,q}(\zeta) \text{ for all } q > \lambda.$$

In all these cases ϕ coincides with its corresponding Padé approximant, that is, $[\phi; k/\lambda]_\zeta(z) \equiv \phi(z)$ and $[\phi; p/\lambda]_\zeta(z) \equiv \phi(z)$ for all $p > k$ and $[\phi; k/q]_\zeta(z) \equiv \phi(z)$ for $q > \lambda$.

The Möbius function $z \rightarrow \frac{z+1}{z-1}$, maps every orthogonal circle to the real axis that passes through 1, to a line parallel to the imaginary axis.

Thus, as z varies in such a circle, $\operatorname{Re}\left(\frac{z+1}{z-1}\right)$ remains constant. This yields that $\left|\exp\left(\frac{z+1}{z-1}\right)\right|$ remains constant too.

More specifically, it can be checked that the unit circle is mapped into the unit circle through the mapping $z \rightarrow \exp\left(\frac{z+1}{z-1}\right)$.

Now consider the mapping g defined on the set $\{z \in \mathbb{C} \mid \operatorname{Re}(z) \leq 1\}$

$$g(z) = \begin{cases} (z-1) \exp\left(\frac{z+1}{z-1}\right), & z \neq 1 \\ 0, & z = 1. \end{cases}$$

Then the following proposition holds;

Proposition 2.3. *There exist a Jordan domain V , subset of the set $S = \{z \in \mathbb{C} \mid \operatorname{Re}(z) \leq 1, \operatorname{Im}(z) \geq 0\}$, with the following properties*

- (i) *V is contained in a set bounded from two arcs that belong in S and are arcs of circles orthogonal to the real axis, passing through 1.*
- (ii) *V contains an open arc of the unit circle that ends at 1 and $1 \in \partial V$.*
- (iii) *The function g defined above is one-to-one in V .*
- (iv) *The function $\frac{1}{\left\{\exp \frac{z+1}{z-1}\right\} \log(1-z)}$ belong to $A(V)$, which means that it is continuous on \overline{V} and holomorphic on V .*

Proof. Consider the arc of the unit circle $A_n = \left\{e^{it} : \frac{1}{n} \leq t \leq \frac{\pi}{2}\right\}$ for $n = 1, 2, \dots$.

Because

$$g'(z) = \exp\left(\frac{z+1}{z-1}\right) \frac{z-3}{z-1} \neq 0, \quad \text{for } z \in S \setminus \{1\},$$

we get that for every $z \in A_n$, there exists $r = r(z) > 0$ such that $g|_{D(z,r)}$ is one-to-one, where $D(z,r) \subset S$.

Thus, because A_n is compact, there are $z_1, z_2, \dots, z_{m_n} \in A_n$ and $r_1, \dots, r_{m_n} > 0$, $m_n \in \mathbb{N}$ such that $A_n \subset \bigcup_{i=1}^{m_n} D(z_i, r_i) \subset S$, and for every $i = 1, 2, \dots, m_n$, $g|_{D(z_i, r_i)}$ is one-to-one and $\frac{\pi}{2} \geq \arg(z_1) > \arg(z_2) > \dots > \arg(z_{m_n}) \geq \frac{1}{n}$.

Take now $w_i \in D(z_i, r_i) \cap D(z_{i+1}, r_{i+1})$, for $i = 1, 2, \dots, m_n - 1$. Define $V_{i,\varepsilon} = \{e^{it}(w-1) + 1 \mid |w| = 1, \arg w_{i+1} \leq \arg w \leq \arg w_i \text{ and } |t| < \varepsilon\}$, $i = 1, 2, \dots, m_n - 1$, where $\varepsilon > 0$ is small enough such that $V_{i,\varepsilon} \subsetneq D(z_{i+1}, r_{i+1})$. Fix $i \in \{1, \dots, m_n - 1\}$ and denote V_i the set $V_{i,\varepsilon}$ for the previous ε depending on i .

Now we claim that there are two arcs of circles orthogonal to the x -axis, that pass through 1, one with radius equal to $a_i < 1$, and one with radius $b_i > 1$ with centers $0 < 1 - a_i < 1$ and $1 - b_i < 0$, such that, if we call $W_{a_i b_i}$ the set of the points between these arcs with argument in $(0, \arg w_1)$ g satisfies the following:

If $z \in W_{a_i, b_i} \cap V_i$ and $\tilde{z} \in W_{a_i, b_i}$ with $g(z) = g(\tilde{z})$, then $z = \tilde{z}$.

For $a_i < 1$ and $b_i > 1$, which will be determined later on, we consider $z \in W_{a_i, b_i} \cap V_i$ and $\tilde{z} \in W_{a_i, b_i}$.

If $g(z) = g(\tilde{z})$, then $\frac{|\tilde{z} - 1|}{|z - 1|} = e^{\operatorname{Re}\left(\frac{z+1}{z-1}\right) - \operatorname{Re}\left(\frac{\tilde{z}+1}{\tilde{z}-1}\right)}$.

Now, because $z, \tilde{z} \in W_{a_i, b_i}$ and the fact that, if a complex number t belongs to a circle orthogonal to the real axis and passes through 1 of radius $r > 0$, then $\operatorname{Re}\left(\frac{t+1}{t-1}\right) = 2 + \frac{1}{r}$; It follows that $e^{\frac{1}{b_i} - \frac{1}{a_i}} \leq \frac{|\tilde{z} - 1|}{|z - 1|} \leq e^{\frac{1}{a_i} - \frac{1}{b_i}}$.

Now if we choose a_i and b_i close enough, this will make \tilde{z} to be inside $D(z_{i+1}, r_{i+1})$, yielding $g(z) = g(\tilde{z})$, where $z, \tilde{z} \in D(z_{i+1}, r_{i+1})$ and thus $z = \tilde{z}$, because g is one to one in $D(z_{i+1}, r_{i+1})$.

By choosing the pairs (a_i, b_i) , $i = 1, 2, \dots, m_n - 2$ to satisfy also $W_{a_{i+1}, b_{i+1}} \subset W_{a_i, b_i}$ we get that g is one-to-one on $\bigcup_{i=1}^{m_n-2} (W_{a_i, b_i} \cap V_i) = S_n$. Moreover, if $z \in S_n$ and $w \in W_{a_{m_n-2}, b_{m_n-2}} \cup S_n$ satisfy $g(z) = g(w)$, then $z = w$.

Carrying this procedure as n goes to infinity by taking the union of S_n , we set $V = \bigcup_{n=1}^{\infty} S_n$ and one can verify that V satisfies all requirements. Especially the standard singular inner function $\exp \frac{z+1}{z-1}$ is far from ∞ and 0 on V .

Thus, we have $\frac{1}{\left[\exp \frac{z+1}{z-1}\right] \log(1-z)} \in A(V)$. \blacksquare

Lemma 2.4. *Let $h : [0, t_0] \rightarrow \mathbb{C}$ be a continuous function on $(0, t_0]$, where $t_0 > 0$. We assume that $\lim_{t \rightarrow 0} \arg(h(t)) = c \in \mathbb{R}$ and $\int_{0^+}^{t_0} |h(t)| dt = +\infty$. Then $\left| \int_{0^+}^{t_0} h(t) dt \right|$ equals also $+\infty$.*

Proof. Let $t_1 > 0$ be such that $t_1 < t_0$ and $|\arg(h(t)) - s| < \frac{\pi}{3}$ for all $0 < t < t_1$.

Then for \tilde{t} , $0 < \tilde{t} < t_1$ it holds

$$\begin{aligned} \left| \int_{\tilde{t}}^{t_1} h(t) dt \right| &= \left| \int_{\tilde{t}}^{t_1} h(t) e^{is} dt \right| = \left| \int_{\tilde{t}}^{t_1} |h(t)| e^{i(\arg(h(t)) - s)} dt \right| \\ &\geq \operatorname{Re} \left(\int_{\tilde{t}}^{t_1} |h(t)| e^{i(\arg(h(t)) - s)} dt \right) \\ &= \int_{\tilde{t}}^{t_1} |h(t)| \cos(\arg(h(t)) - s) dt \end{aligned}$$

which is bigger than $\frac{1}{2} \int_{\tilde{t}}^{t_1} |h(t)| dt$. Since $\int_{0^+}^{t_0} |h(t)| dt = +\infty$, it follows easily that

$$\left| \int_{0^+}^{t_1} h(t) dt \right| = +\infty.$$

Moreover, the last implies that, $\left| \int_{0^+}^{t_0} h(t) dt \right| = +\infty$, because h is continuous on $(0, t_0]$.
The proof is complete. ■

3 Smooth Universal Padé Approximants

For the definitions of $X^\infty(\Omega)$ and the notion of the Padé Approximants we refer to § 2 and we state the following.

Theorem 3.1. *Let $F \subset \mathbb{N} \times \mathbb{N}$ be a set that contains a sequence $(\tilde{p}_n, \tilde{q}_n)$, $n = 1, 2, \dots$, such that $\tilde{p}_n \rightarrow +\infty$ and $\tilde{q}_n \rightarrow +\infty$ and let $\Omega \subseteq \mathbb{C}$ an open set. Let $L, \Delta \subset \mathbb{C}$ be compact sets inside $\overline{\Omega}$ and K a compact set in \mathbb{C} such that $K \cap \overline{\Omega} = \emptyset$.*

Then there exists $f \in X^\infty(\Omega)$ such that: for every rational function h there exists a sequence $(p_n, q_n) \in F$ ($n = 1, 2, \dots$) with the following properties:

- (i) $f \in D_{p_n, q_n}(\zeta) \cap E_{p_n, q_n, \zeta}(K \cup \Delta)$, for every $\zeta \in L$.
- (ii) For every $\ell \in \mathbb{N}$, $\sup_{\zeta \in L} \sup_{z \in \Delta} \left| [f; p_n/q_n]_\zeta^{(\ell)}(z) - f^{(\ell)}(z) \right| \rightarrow 0$, as $n \rightarrow +\infty$.
- (iii) $\sup_{\zeta \in L} \sup_{z \in K} \chi([f; p_n/q_n]_\zeta(z), h(z)) \rightarrow 0$, as $n \rightarrow +\infty$.

The set of such functions $f \in X^\infty(\Omega)$ is dense and G_δ in $X^\infty(\Omega)$.

Proof. Let $(f_i)_{i \in \mathbb{N}}$ be an enumeration of the rational functions with coefficients of the numerator and the denominator from $\mathbb{Q} + i\mathbb{Q}$.

We name \mathcal{U} the set of all functions in $X^\infty(\Omega)$ that satisfy the properties (i), (ii) and (iii), and we will prove that \mathcal{U} is a G_δ -dense in the $X^\infty(\Omega)$ -topology and therefore, $\mathcal{U} \neq \emptyset$.

For $j, s \in \mathbb{N}^*$ and $(p, q) \in F$ we define:

$$E(j, p, q, s) = \left\{ f \in X^\infty(\Omega) \mid f \in D_{p, q}(\zeta) \cap E_{p, q, \zeta}(K) \text{ for all } \zeta \in L \text{ and } \sup_{\zeta \in L} \sup_{z \in K} \chi([f; p/q]_\zeta(z), f_j(z)) < \frac{1}{s} \right\}$$

and,

$$T(p, q, s) = \{ f \in X^\infty(\Omega) \mid f \in D_{p, q}(\zeta) \cap E_{p, q, \zeta}(\Delta) \text{ for all } \zeta \in L \text{ and } \sup_{\zeta \in L} \sup_{z \in \Delta} \left| [f; p/q]_\zeta^{(\ell)}(z) - f^{(\ell)}(z) \right| < \frac{1}{s} \text{ for } \ell = 0, 1, \dots, s \}.$$

Proposition 2.1 and the definition of $X^\infty(\Omega)$ imply that

$$U = \bigcap_{j,s=1}^{\infty} \bigcup_{(p,q) \in F} (E(j,p,q,s) \cap T(p,q,s)).$$

To prove that U is a G_δ -dense in the $X^\infty(\Omega)$ -topology, it is enough to prove that for every $j, s = 1, 2, \dots$ and $(p, q) \in F$ the sets $E(j, p, q, s)$, $T(p, q, s)$ are open in $X^\infty(\Omega)$ and that for every j and s inside \mathbb{N}^* , the set $\bigcup_{(p,q) \in F} (E(j, p, q, s) \cap T(p, q, s))$ is dense in $X^\infty(\Omega)$.

Now let $j, s \in \mathbb{N}^*$ and $(p, q) \in F$. We first prove that the set $E(j, p, q, s)$ is open in $X^\infty(\Omega)$. Indeed, let $f \in E(j, p, q, s)$ and let $g \in X^\infty(\Omega)$ be such that,

$$\sup_{z \in L} |f^{(m)}(z) - g^{(m)}(z)| < a \quad \text{for } m = 0, 1, 2, \dots, p + q + 1 \quad (1)$$

The number $a > 0$ will be determined later on. It is enough to prove that if a is small enough then $g \in E(j, p, q, s)$.

The Hankel determinants defining $D_{p,q}(\zeta)$ for f depend continuously on $\zeta \in L$; thus, there exists $\delta > 0$ such that the absolute values of the corresponding Hankel determinants are greater than $\delta > 0$, for every $\zeta \in L$, because $f \in D_{p,q}(\zeta)$ for every $\zeta \in L$ and because L is compact.

From (1) we can control the first $p + q + 1$ Taylor coefficients of g and by making $a > 0$ small enough one can get the Hankel determinants that define $D_{p,q}(\zeta)$ to have absolute value at least $\delta/2 > 0$.

Therefore, g will belong in $D_{p,q}(\zeta)$ for every $\zeta \in L$. Now we consider the Padé approximants of f, g according to the Jacobi formula (see preliminaries)

$$[f; p/q]_\zeta(z) = \frac{A(f, \zeta)(z)}{B(f, \zeta)(z)} \quad \text{and} \quad [g; p/q]_\zeta(z) = \frac{A(g, \zeta)(z)}{B(g, \zeta)(z)}.$$

Now $|A(f, \zeta)(z)|^2 + |B(f, \zeta)(z)|^2$ vary continuously with respect to $(z, \zeta) \in K \times L$, because of the Jacobi formula. So, there is a $\delta' > 0$ such that:

$$|A(f, \zeta)(z)|^2 + |B(f, \zeta)(z)|^2 \geq \delta', \quad \text{for all } \zeta \in L \text{ and } z \in K.$$

Now again from the Jacobi formula, if a is small enough, one gets:

$$|A(g, \zeta)(z)|^2 + |B(g, \zeta)(z)|^2 \geq \delta'/2, \quad \text{for all } \zeta \in L \text{ and } z \in K.$$

This yields that $g \in E_{p,q,\zeta}(K)$ for every $\zeta \in L$. For the rest it is enough to show that if a is small enough then $\sup_{\zeta \in L} \sup_{z \in K} \chi([g; p/q]_\zeta(z), [f; p/q]_\zeta(z))$ can become less than

$\frac{1}{s} - \sup_{\zeta \in L} \sup_{z \in K} \chi([f; p/q]_{\zeta}(z), f_j(z)) \equiv \gamma > 0$. By taking a small as before we have that $|A(f, \zeta)(z)|^2 + |B(f, \zeta)(z)|^2 > \delta'$ and for every $\zeta \in L$ and $z \in K$ we have $|A(g, \zeta)(z)|^2 + |B(g, \zeta)(z)|^2 > \delta'/2$.

It follows that

$$\begin{aligned} & \chi([f; p/q]_{\zeta}(x), [g; p/q]_{\zeta}(z)) \\ &= \frac{|A(f, \zeta)(z)B(g, \zeta)(z) - A(g, \zeta)(z)B(f, \zeta)(z)|}{\sqrt{|A(f, \zeta)(z)|^2 + |B(f, \zeta)(z)|^2} \sqrt{|A(g, \zeta)(z)|^2 + |B(g, \zeta)(z)|^2}} \\ &\leq \frac{\sqrt{2}}{\delta'} |A(f, \zeta)(z)B(g, \zeta)(z) - A(g, \zeta)(z)B(f, \zeta)(z)| \end{aligned}$$

for all $\zeta \in L$ and $z \in K$, which easily yields the result, because the last expression can become as small as we want to, uniformly for all $\zeta \in L, z \in K$. Thus, we proved that $E(j, p, q, s)$ is open.

Next, we prove that $T(p, q, s)$ is also open in $X^{\infty}(\Omega)$. Let f be a function inside $T(p, q, s)$, L' be a compact set inside $\overline{\Omega}$ such that $L' \supset L \cup \Delta$ and let g be a function inside $X^{\infty}(\Omega)$ such that: $\sup_{z \in L'} |f^{(m)}(z) - g^{(m)}(z)| < a$, for $m = 0, 1, 2, \dots, \max(s, p + q + 1)$, where $a > 0$ will be determined later on.

In the same way as before one deduce that by making “ a ” small enough it follows $g \in D_{p,q}(\zeta) \cap E_{p,q,\zeta}(\Delta)$, for all $\zeta \in L$.

Now $f(z) \in \mathbb{C}$, for each $z \in \Delta$. It follows that for all $\zeta \in L, z \in \Delta$ we have $[f; p/q]_{\zeta}(z) \in \mathbb{C}$. Therefore, $B(f, \zeta)(z) \neq 0$, where B is given by the Jacobi formula.

So there is a $\delta'' > 0$ such that $\delta'' < 1$ and $|B(f, \zeta)(z)| > \delta''$ for all $\zeta \in L, z \in \Delta$, because $L \times \Delta$ is compact.

By making “ a ” small enough, by continuity one can get

$$|B(g, \zeta)(z)| > \frac{\delta''}{2} \quad \text{for all } \zeta \in L, z \in \Delta.$$

For $\ell \in \{0, 1, \dots, s\}$ it holds

$$\begin{aligned} \sup_{\zeta \in L} \sup_{z \in \Delta} |[g; p/q]_{\zeta}^{(\ell)}(z) - g^{(\ell)}(z)| &\leq \sup_{z \in \Delta} |f^{(\ell)}(z) - g^{(\ell)}(z)| \\ &+ \sup_{\zeta \in L} \sup_{z \in \Delta} |f^{(\ell)}(z) - [f; p/q]_{\zeta}^{(\ell)}(z)| \\ &+ \sup_{\zeta \in L} \sup_{z \in \Delta} |[g; p/q]_{\zeta}^{(\ell)}(z) - [f; p/q]_{\zeta}^{(\ell)}(z)|. \end{aligned}$$

The first term obviously get small as “ a ” gets small, because $L' \supset \Delta$. Since the second term is fixed and less than $1/s$ we must control only the last term.

But the Jacobi denominators of $[f; p/q]_{\zeta}^{(\ell)}(z)$ and $[g; p/q]_{\zeta}^{(\ell)}(z)$ are bounded below from $(\delta'')^{\ell+1}$ and $(\delta''/2)^{\ell+1}$ respectively for $\ell = 0, 1, \dots, s$.

Thus, the last term can get as small as we want to for all $\ell = 0, 1, \dots, s$, if a is small enough. We are done.

Finally, we prove that for all $j, s \in \mathbb{N}$ the set $\bigcup_{(p,q) \in F} (E(j, p, q, s) \cap T(p, q, s))$ is dense in $X^{\infty}(\Omega)$.

Let $L' \subset \mathbb{C}$ be a compact set inside $\overline{\Omega}$ such that $L \cup \Delta \subset L'$. Without loss of generality we may assume that every connected component of $\tilde{\mathbb{C}} \setminus L'$ contains a point that belongs to $\tilde{\mathbb{C}} \setminus \overline{\Omega}$. This can be achieved, for example, by taking $L' = \overline{\Omega} \cap \overline{D(0, n)}$, for big enough $n \in \mathbb{N}$.

Let also g be a function inside $X^{\infty}(\Omega)$, $N \in \mathbb{N}$ and $\varepsilon > 0$. We can assume, without loss of generality, that g is a rational function with poles off $\overline{\Omega}$, because of the definition of $X^{\infty}(\Omega)$.

To prove what we want to, we have to find a function $f \in X^{\infty}(\Omega)$ and a pair $(p, q) \in F$ such that:

- (i) $f \in D_{p,q}(\zeta) \cap E_{p,q,\zeta}(K \cup \Delta)$, for all $\zeta \in L$.
- (ii) $\sup_{\zeta \in L} \sup_{z \in K} \chi([f; p/q]_{\zeta}(z), f_j(z)) < \frac{1}{s}$.
- (iii) $\sup_{\zeta \in L} \sup_{z \in \Delta} |f^{(\ell)}(z) - [f; p/q]_{\zeta}^{(\ell)}(z)| < \frac{1}{s}$, for $\ell = 0, 1, \dots, s$
- (iv) $\sup_{z \in L'} |f^{(m)}(z) - g^{(m)}(z)| < \varepsilon$, for $m = 0, 1, \dots, N$.

$$\text{Let } \omega : L' \cup K \rightarrow \mathbb{C}, \text{ such that } \omega(z) = \begin{cases} f_j(z), & z \in K \\ g(z), & z \in L' \end{cases}.$$

Now, let μ be the sum of the principal parts of the poles of the rational function f_j that belong to K . Then $(\omega - \mu)$ is holomorphic in a neighborhood of $(L' \cup K)$. Combining Runge's with Weierstrass Theorems we conclude that there exists a rational function $\frac{\tilde{A}(z)}{\tilde{B}(z)}$ with poles out of $(L' \cup K)$, approximating $(\omega - \mu)$ uniformly on $L' \cup K$ with respect to the euclidean metric and in the level of all derivatives of order from zero to N . That implies that the function $\frac{A(z)}{B(z)} = \mu(z) + \frac{\tilde{A}(z)}{\tilde{B}(z)}$ approximates $f_j(z)$, uniformly on K with respect to the chordal distance, and also that $\left(\frac{A(z)}{B(z)}\right)^{(\ell)}$ approximates the function $(g(z))^{(\ell)}$ uniformly on L' , with respect to the euclidean metric. Obviously, we can assume that the greatest common divisor of $A(z)$ and $B(z)$ is equal to one.

From our assumption on F , there exists a pair $(p, q) \in F$ such that $p > \deg A, \deg B$ and $q > \deg B$. We consider the function $\frac{A(z)}{B(z)} + dz^T = \frac{A(z) + d \cdot z^T \cdot B(z)}{B(z)}$ where $T = p - \deg B$ and d is different than zero. Now, it is easy to see that $\gcd(A(z) + dz^T B(z), B(z))$ equals again to one. Thus, according to Proposition 2.2 it holds that for all $\zeta \in \mathbb{C}$ such that $B(\zeta) \neq 0$ the rational function $\frac{A(z) + dz^T B(z)}{B(z)}$ belongs to $D_{p,q}(\zeta)$ and also $\left[\frac{A(z) + dz^T B(z)}{B(z)}; p/q \right]_{\zeta}(z) = \frac{A(z) + dz^T B(z)}{B(z)}$. In particular the above hold for all $\zeta \in L'$, because $B(\zeta) \neq 0$ for all $\zeta \in L'$.

We distinguish the cases $B(z) \neq 0$ for all $z \in \overline{\Omega}$ and the case where B has roots in $\overline{\Omega} \setminus L'$. First assume that $B(z) \neq 0$ for all $z \in \overline{\Omega}$. In this case we set $f(z) = \frac{A(z)}{B(z)} + dz^T$, and by selecting d with $|d|$ small enough, we are done.

In the second case, since every component of $\tilde{\mathbb{C}} \setminus L'$ contains a point from $\tilde{\mathbb{C}} \setminus \overline{\Omega}$, there exists a rational function that belongs to $X^\infty(\Omega)$, call it f , such that every finite set of derivatives $f^{(\ell)}$ are close to $\left(\frac{A(z) + dz^T}{B(z)} \right)^{(\ell)}$ uniformly on L' . This is immediate from Runge's and Weierstrass Theorems and also from the fact that B has finitely many roots outside L' and thus in a positive distance from L' .

It is easy to see that f fulfills all requirements in the same way as $\frac{A(z) + dz^T B(z)}{B(z)}$ does except from the fact that maybe $[f; p/q]_{\zeta}(z) \neq f(z)$.

But the following is true:

$$\begin{aligned} \sup_{\zeta \in L} \sup_{z \in \Delta} | [f; p/q]_{\zeta}^{(\ell)}(z) - f^{(\ell)}(z) | &\leq \sup_{z \in \Delta} | f^{(\ell)}(z) - h^{(\ell)}(z) | \\ &+ \sup_{\zeta \in L} \sup_{z \in \Delta} | h^{(\ell)}(z) - [h; p/q]_{\zeta}^{(\ell)}(z) | \\ &+ \sup_{\zeta \in L} \sup_{z \in \Delta} | [h; p/q]_{\zeta}^{(\ell)}(z) - [f; p/q]_{\zeta}^{(\ell)}(z) |. \quad (*) \end{aligned}$$

with $h(z) = \frac{A(z) + dz^T B(z)}{B(z)}$.

Now, as p, q are fixed and we can control any finite set of derivatives of f , we can also control any finite set of Taylor coefficients of f . Thus, we can make the first and the last term of the right-hand side expression in $(*)$ as small as we want to and we are done.

This completes the proof of the Theorem. ■

If we set $K = \emptyset$ and $L = \Delta = L_n$ where $L_n = \overline{\Omega} \cap \overline{D(0, n)}$ for $n = 1, 2, \dots$ and we apply Baire's Theorem once more, we obtain the result that generically all $f \in X^\infty(\Omega)$

can be approximated by their Padé approximants $[f; p_n/q_n]_\zeta(z)$, $(p_n, q_n) \in F$, provided that $F \subset \mathbb{N} \times \mathbb{N}$ contains a sequence $(\tilde{p}_n, \tilde{q}_n) \in F$, $n = 1, 2, \dots$ such that $\tilde{p}_n \rightarrow +\infty$ and $\tilde{q}_n \rightarrow +\infty$.

If we set $L = \{\zeta\} \subset \overline{\Omega}$, $K = K_n$, $\Delta = L_n$ where K_n is an exhausting sequence of $\mathbb{C} \setminus \overline{\Omega}$ [18] and $L_n = \overline{\Omega} \cap \overline{D(0, n)}$ for $n = 1, 2, \dots$ and then apply Baire's Theorem, provided that the set $F \subset \mathbb{N} \times \mathbb{N}$ contains a sequence $((\tilde{p}_n, \tilde{q}_n))_{n \in \mathbb{N}}$ where $\tilde{p}_n \rightarrow +\infty$ and $\tilde{q}_n \rightarrow +\infty$ we obtain the following:

Theorem 3.2. *Let $\Omega \subset \mathbb{C}$ be an open set and $\zeta \in \overline{\Omega}$ be fixed. Then there exist $f \in X^\infty(\Omega)$ such that, for every rational function h and every compact set $K \subset \mathbb{C} \setminus \overline{\Omega}$ there exists a sequence $(p_n, q_n) \in F$, $n = 1, 2, \dots$ such that $f \in D_{p_n, q_n}(\zeta)$ for all $n \in \mathbb{N}$, and $\sup_{z \in K} \chi([f; p_n/q_n]_\zeta(z), h(z)) \rightarrow 0$ and for every compact set $L' \subset \overline{\Omega}$ there is a $n(L') \in \mathbb{N}$ such that $f \in E_{p_n, q_n, \zeta}(K \cup L')$, for all $n \geq n(L')$ and also $\sup_{z \in L'} |[f; p_n/q_n]_\zeta^{(\ell)}(z) - f^{(\ell)}(z)| \rightarrow 0$, as $n \rightarrow +\infty$, for all $\ell \in \mathbb{N}$. The set of such functions $f \in X^\infty(\Omega)$ is dense and G_δ in $X^\infty(\Omega)$.*

If we set $L = \Delta = L_n$ and $K = K_n$ for $n = 1, 2, \dots$ where $L_n = \overline{\Omega} \cap \overline{D(0, n)}$ and $(K_n)_{n=1}^\infty$ is an exhausting sequence of $\mathbb{C} \setminus \overline{\Omega}$ by applying Baire Theorem we obtain the following:

Theorem 3.3. *Let F be a subset of $\mathbb{N} \times \mathbb{N}$ containing a sequence $(\tilde{p}_n, \tilde{q}_n) \in F$, $n = 1, 2, \dots$ with $\tilde{p}_n \rightarrow +\infty$, $\tilde{q}_n \rightarrow +\infty$. Let $\Omega \subset \mathbb{C}$ be an open set. Then there exists a function $f \in X^\infty(\Omega)$ that satisfy the following:*

For every compact set $K \subset \mathbb{C} \setminus \overline{\Omega}$ and rational function h there exists a sequence $(p_n, q_n) \in F$, $n = 1, 2, \dots$ such that the following hold:

For every compact set $L \subset \overline{\Omega}$ there exists a $n(L) \in \mathbb{N}$ such that

$$f \in D_{p_n, q_n}(\zeta), \text{ for all } n \geq n(L) \text{ and } \zeta \in L$$

$$f \in E_{p_n, q_n, \zeta}(K \cup L), \text{ for all } n \geq n(L) \text{ and } \zeta \in L$$

and

$$\sup_{\zeta \in L} \sup_{z \in K} \chi([f; p_n/q_n]_\zeta(z), h(z)) \rightarrow 0, \text{ as } n \rightarrow +\infty$$

$$\sup_{\zeta \in L} \sup_{z \in L} |[f; p_n/q_n]_\zeta^{(\ell)}(z) - f^{(\ell)}(z)| \rightarrow 0, \text{ as } n \rightarrow +\infty \text{ for all } \ell = 0, 1, 2, \dots$$

The set of such functions $f \in X^\infty(\Omega)$ is dense and G_δ in $X^\infty(\Omega)$.

4 The case $\{\infty\} \cup (\mathbb{C} \setminus \overline{\Omega})$ connected

We recall that if $K \subseteq \mathbb{C}$ is compact then $A(K) = \{h : K \rightarrow \mathbb{C} \text{ continuous on } K \text{ and holomorphic in } K^0\}$.

Theorem 4.1. *Let $F \subset \mathbb{N} \times \mathbb{N}$ be a set that contains a sequence $(\tilde{p}_n \tilde{q}_n)$, $n = 1, 2, \dots$ such that $\tilde{p}_n \rightarrow +\infty$ and $\Omega \subseteq \mathbb{C}$ an open set such that $\{\infty\} \cup (\mathbb{C} \setminus \overline{\Omega})$ is connected.*

Let $L, \Delta \subset \mathbb{C}$ compact sets inside $\overline{\Omega}$ and K be a compact set in \mathbb{C} such that K^c is connected and $K \cap \overline{\Omega} = \emptyset$. Then there exist $f \in X^\infty(\Omega)$ such that:

For every function h in $A(K)$ there exists a sequence $(p_n, q_n) \in F$, $n = 1, 2, \dots$ such that:

$$(i) \ f \in D_{p,q}(\zeta) \cap E_{p,q,\zeta}(K \cup \Delta), \text{ for all } \zeta \in L$$

$$(ii) \text{ for all } \ell = 0, 1, 2, \dots, \sup_{\zeta \in L} \sup_{z \in \Delta} | [f; p_n/q_n]_\zeta^{(\ell)}(z) - f^{(\ell)}(z) | \rightarrow 0 \text{ as } n \rightarrow +\infty$$

$$(iii) \sup_{\zeta \in L} \sup_{z \in K} | [f; p_n/q_n]_\zeta(z) - h(z) | \rightarrow 0, \text{ as } n \rightarrow +\infty.$$

The set of such functions $f \in X^\infty(\Omega)$ is dense and G_δ is $C^\infty(\Omega)$.

Proof. Let $(f_j)_{j=1}^\infty$ be an enumeration of all polynomial functions with coefficients from $\mathbb{Q} + i\mathbb{Q}$.

We name \mathcal{U} the set of the functions with the properties (i), (ii), (iii) and we will prove that \mathcal{U} is G_δ -dense set in the $X^\infty(\Omega)$ -topology and so $\mathcal{U} \neq \emptyset$.

For $j, s \in \mathbb{N}^*$ and $(p, q) \in F$ we define:

$$E(j, p, q, s) = \{f \in X^\infty(\Omega) \mid f \in D_{p,q}(\zeta) \cap E_{p,q,\zeta}(K), \text{ for all } \zeta \in L \text{ and } \sup_{\zeta \in L} \sup_{z \in K} |[f; p/q]_\zeta(z) - f_j(z)| < \frac{1}{s}\}$$

$$T(p, q, s) = \{f \in X^\infty(\Omega) \mid f \in D_{p,q}(\zeta) \cap E_{p,q,\zeta}(\Delta), \text{ for all } \zeta \in L \text{ and } \sup_{\zeta \in L} \sup_{z \in \Delta} |[f; p/q]_\zeta^{(\ell)}(z) - f^{(\ell)}(z)| < \frac{1}{s} \text{ for } \ell = 0, 1, \dots, s\}.$$

It is true that $\mathcal{U} = \bigcap_{j,s=1}^\infty \bigcup_{(p,q) \in F} (E(j, p, q, s) \cap T(p, q, s))$. This can easily be verified using Mergelyans' Theorem. Now to prove that \mathcal{U} is a G_δ -dense set in $X^\infty(\Omega)$, it is enough to show that for every $j, s = 1, 2, \dots$ and $(p, q) \in F$ the sets $E(j, p, q, s)$ and $T(p, q, s)$ are

open in $X^\infty(\Omega)$ and that for every $j, s = 1, 2, \dots$, the set $\bigcup_{(p,q) \in F} (E(j, p, q, s) \cap T(p, q, s))$ is dense in $X^\infty(\Omega)$.

So, let $j, s \in \mathbb{N}^*$ and a pair $(p, q) \in F$. We first prove that $E(j, p, q, s)$ is open in $X^\infty(\Omega)$.

Indeed, let $f \in E(j, p, q, s)$, and let $g \in X^\infty(\Omega)$ be such that

$$\sup_{z \in L} |f^{(m)}(z) - g^{(m)}(z)| < a \quad \text{for } m = 0, 1, 2, \dots, p + q + 1 \quad (2)$$

The number $a > 0$ will be determined later on. It is enough to prove that if a is small enough then $g \in E(j, p, q, s)$.

The Hankel determinants defining $D_{p,q}(\zeta)$ for f depend continuously on $\zeta \in L$; thus, there exists $\delta > 0$ such that the absolute values of the corresponding Hankel determinants are greater than $\delta > 0$, for every $\zeta \in L$, because $f \in D_{p,q}(\zeta)$, for every $\zeta \in L$ and because L is compact.

From (1) we can control the first $p + q + 1$ Taylor coefficients of g and by making $a > 0$ small enough one can get the Hankel determinants that define $D_{p,q}(\zeta)$ to have absolute value at least $\delta/2 > 0$.

Therefore, g will belong in $D_{p,q}(\zeta)$ for every $\zeta \in L$. Now we consider the Padé approximants of f, g according to the Jacobi formula (see preliminaries)

$$[f, p/q]_\zeta(z) = \frac{A(f, \zeta)(z)}{B(f, \zeta)(z)} \quad \text{and} \quad [g, p/q]_\zeta(z) = \frac{A(g, \zeta)(z)}{B(g, \zeta)(z)}.$$

Now $|A(f, \zeta)(z)|^2 + |B(f, \zeta)(z)|^2$ vary continuously with respect to $(z, \zeta) \in K \times L$, because of the Jacobi formula. So, there is a $\delta' > 0$ such that:

$$|A(f, \zeta)(z)|^2 + |B(f, \zeta)(z)|^2 \geq \delta', \quad \text{for all } \zeta \in L \text{ and } z \in K.$$

Now again from the Jacobi formula, if a is small enough, one gets:

$$|A(g, \zeta)(z)|^2 + |B(g, \zeta)(z)|^2 \geq \delta'/2, \quad \text{for all } \zeta \in L \text{ and } z \in K.$$

This yields that $g \in E_{p,q,\zeta}(K)$ for every $\zeta \in L$.

Now because $|[f, p/q]_\zeta(z) - f_j(z)| < \frac{1}{s}$, for all $\zeta \in L, z \in K$ it follows that $[f, p/q]_\zeta(z) \in \mathbb{C}$, for all $\zeta \in L, z \in K$.

Thus, there exist $\delta'' > 0$ such that $|B(f, \zeta)(z)| > \delta''$ for all $\zeta \in L, z \in K$, because $L \times K$ is compact. Because the first $p + q + 1$ Taylor coefficients of g can be controlled and because of the Jacobi formula, by making “ a ” small enough, one gets

$$|B(g, \zeta)(z)| > \frac{\delta''}{2}, \quad \text{for all } \zeta \in L, z \in K.$$

To complete the proof that $E(j, p, q, s)$ is open, it is enough to show that $\sup_{\zeta \in L} \sup_{z \in K} | [g; r/q]_{\zeta}(z) - [f; p/q]_{\zeta}(z) |$ can become less than $\frac{1}{s} - \sup_{\zeta \in L} \sup_{z \in K} | [f; p/q]_{\zeta}(z) - f_j(z) | = \gamma > 0$.

But

$$\sup_{\zeta \in L} \sup_{z \in K} | [f; p/q]_{\zeta}(z) - [g; p/q]_{\zeta}(z) | \leq \frac{2}{(\delta^n)^2} | A(f, \zeta)B(g, \zeta)(z) - B(f, \zeta)(z)A(g, \zeta)(z) |.$$

This easily yields the result because the expression on the right-hand side of the inequality can become as small as we want to, for $\ell = 0, 1, \dots, s$.

The proof that $T(p, q, s)$ is open in $X^\infty(\Omega)$ is similar to the corresponding proof in Theorem 3.1 and is omitted.

Finally, we prove that for every $j, s \in \mathbb{N}^*$ the set $\bigcup_{(p,q) \in F} (E(j, p, q, s) \cap T(p, q, s))$ is dense in $X^\infty(\Omega)$. Let $L' \subset \mathbb{C}$ be a compact set inside $\overline{\Omega}$, such that $L \cup \Delta \subset L'$. We can assume, without loss of generality, that $L' = \overline{\Omega} \cap \overline{D(0, n)}$ for some $n \in \mathbb{N}$, big enough. Let g be a function inside $X^\infty(\Omega)$, $N \in \mathbb{N}$ and $\varepsilon > 0$. We can assume by the definition of $X^\infty(\Omega)$ and from the fact that $\{\infty\} \cup (\mathbb{C} \setminus \overline{\Omega})$ is connected, that g is a polynomial. We have to find a function f inside $X^\infty(\Omega)$ and a pair $(p, q) \in F$ such that:

- (i) $f \in D_{p,q}(\zeta) \cap E_{p,q,\zeta}(K \cup \Delta)$, for all $\zeta \in L$
- (ii) $\sup_{\zeta \in L} \sup_{z \in \Delta} | [f; p/q]_{\zeta}^{(\ell)}(z) - f^{(\ell)}(z) | < \frac{1}{s}$, for all $\ell = 0, 1, \dots, s$
- (iii) $\sup_{\zeta \in L} \sup_{z \in K} | [f; p/q]_{\zeta}(z) - f_j(z) | < \frac{1}{s}$
- (iv) $\sup_{z \in L'} | f^{(m)}(z) - g^{(m)}(z) | < \varepsilon$ for $m = 0, 1, \dots, N$.

Now, $\tilde{\mathbb{C}} \setminus L' = \tilde{\mathbb{C}} \setminus (\overline{\Omega} \cap \overline{D(0, n)}) = (\{\infty\} \cup (\mathbb{C} \setminus \overline{\Omega})) \cup (\tilde{\mathbb{C}} \setminus \overline{D(0, n)})$ is a connected set as a union of two connected subsets of $\tilde{\mathbb{C}}$, intersecting at least at the point ∞ .

From our hypothesis K^c is connected too. So, as $L \cap K = \emptyset$, there exist two simply connected domains G_1, G_2 such that $G_1 \cap G_2 = \emptyset$, $L \subset G_1$, $K \subset G_2$. We may assume also that G_1, G_2 have positive distance.

We consider now the function $w : G_1 \cup G_2 \rightarrow \mathbb{C}$ with: $w(z) = \begin{cases} f_j(z), & z \in G_2 \\ g(z), & z \in G_1 \end{cases}$. By Runge's theorem there exists a sequence of polynomials \tilde{p}_n that approximate uniformly on compact sets the analytic function w .

Because $G_1 \cup G_2$ is open, according to Weierstrass theorem the approximation will be valid in the level of all derivatives. Therefore, one such polynomial \tilde{p} approximates

f_j uniformly on K with respect to the euclidean distance and $\tilde{p}^{(\ell)}$ approximate $g^{(\ell)}$ with respect to the euclidean metric, uniformly on L' for all $\ell = 0, 1, \dots, N$.

Now, there exists $(p, q) \in F$ with $p > \deg \tilde{p}$, $q \geq 0$ and because $\deg(\tilde{p}(z) + dz^p) = p$, for all $d > 0$ by Proposition 2.2 we have $\tilde{p}(z) + dz^p \in D_{p,q}(\zeta)$ and $[\tilde{p}(z) + dz^p; p/q]_\zeta(z) = \tilde{p}(z) + dz^p$ for all $\zeta \in \mathbb{C}$. But $\tilde{p}(z) + dz^p$ approximate, as $d \rightarrow 0$, the polynomial $\tilde{p}(z)$ uniformly for any finite set of derivatives and on any compact subset of \mathbb{C} .

Therefore, if we choose d sufficiently small and set $f(z) = \tilde{p}(z) + dz^p$, we are done. ■

Varying L , Δ and K we can obtain more complete versions of Theorem 4.1 as we do in Section 3 for Theorem 3.1.

5 Density of rational functions

In this section we give sufficient conditions so that $X^\infty(\Omega) = A^\infty(\Omega)$.

Theorem 5.1. *Let Ω be a bounded, connected and open set such that:*

- (a) $(\overline{\Omega})^0 = \Omega$.
- (b) $\mathbb{C} \setminus \overline{\Omega}$ is connected.
- (c) *There exists $M < +\infty$, such that for every $a, b \in \Omega$ there exists a curve γ inside Ω (i.e. $\gamma : [0, 1] \rightarrow \Omega$ continuous function) such that $\gamma(0) = a$, $\gamma(1) = b$ and $\text{Length}(\gamma) \leq M$.*

Then the polynomials are dense in $A^\infty(\Omega)$ (and therefore $X^\infty(\Omega) = A^\infty(\Omega)$).

Proof. Let $f \in A^\infty(\Omega)$, $\varepsilon > 0$ and $n \in \mathbb{N}_0 = \{0, 1, 2, \dots\}$.

It suffices to find a polynomial p such that:

$$\sup_{z \in \Omega} |f^{(\ell)}(z) - p^{(\ell)}(z)| < \varepsilon, \quad \text{for } \ell = 0, 1, \dots, n.$$

Now $f^{(n)} \in C(\overline{\Omega})$ and is analytic in $\Omega = (\overline{\Omega})^0$, because $f \in A^\infty(\Omega)$. Also $\overline{\Omega}$ is a compact set as Ω is bounded.

Thus, by Mergelyans' Theorem there exists a polynomial, p_n such that $\sup_{z \in \Omega} |f^{(n)}(z) - p_n(z)| < \frac{\varepsilon}{(M+1)^n}$.

Now, fix $z_0 \in \Omega$. Then, for every $z \in \Omega$, there exists a curve γ_z inside Ω that starts at z_0 and ends at z and has length at most M .

Also, for $0 \leq k \leq n-1$, we define the polynomial $p_k(z)$ by:

$$p_k(z) = f^{(k)}(z_0) + \int_{[z_0, z]} p_{k+1}(\zeta) d\zeta$$

and we set $p = p_0$. Then it is obvious that $p^{(k)} = p_0^{(k)} = p_k$, for all k , $0 \leq k \leq n$.

Now for $k = n$ we have:

$$\begin{aligned} \sup_{z \in \Omega} |f^{(k)}(z) - p^{(k)}(z)| &= \sup_{z \in \Omega} |f^{(n)}(z) - p_n(z)| < \frac{\varepsilon}{(M+1)^n} = \frac{e}{(M+1)^k}. \\ \text{Therefore, } \sup_{z \in \Omega} |f^{(k)}(z) - p^{(k)}(z)| &< \frac{\varepsilon}{(M+1)^k}. \end{aligned} \quad (*)$$

Assume that the above relationship holds for a fixed k , $1 \leq k \leq n$. We will prove it for $k-1$: It is:

$$\begin{aligned} \sup_{z \in \Omega} |f^{(k-1)}(z) - p^{(k-1)}(z)| &= \sup_{z \in \Omega} \left| \int_{\gamma_z} (f^{(k)}(\zeta) - p^{(k)}(\zeta)) d\zeta \right| \\ &\leq \sup_{z \in \Omega} \int_{\gamma_z} |f^{(k)}(\zeta) - p^{(k)}(\zeta)| |d\zeta| \\ &\leq \sup_{z \in \Omega} \int_{\gamma_z} \frac{\varepsilon}{(M+1)^k} |d\zeta| \leq \frac{M\varepsilon}{(M+1)^k} < \frac{\varepsilon}{(M+1)^{k-1}}. \end{aligned}$$

which is exactly what we wanted.

That means that $(*)$ is true for all $k = 0, 1, \dots, n$ and our proof is complete. \blacksquare

Theorem 5.2. *Let Ω be a connected, open set such that*

- (a) $(\overline{\Omega})^0 = \Omega$.
- (b) $\{\infty\} \cup (\mathbb{C} \setminus \overline{\Omega})$ is a connected set
- (c) There exists $n_0 \in \mathbb{N}$ such that for every $N \geq n_0$, there exists $M_N > 0$ such that for all $a, b \in \overline{\Omega \cap D(0, N)}^0$ there exists a continuous function $\gamma : [0, 1] \rightarrow \overline{\Omega \cap D(0, N)}^0$ with $\gamma(0) = a$, $\gamma(1) = b$ and $\text{Length}(\gamma) \leq M_N$.

Then the polynomials are dense in $A^\infty(\Omega)$, and therefore $X^\infty(\Omega) = A^\infty(\Omega)$.

For the proof we need two lemmas.

Lemma 5.3. *Let Ω be an open set in \mathbb{C} , such that $\{\infty\} \cup (\mathbb{C} \setminus \overline{\Omega})$ is connected. Then, for every $N \in \mathbb{N}$, $\{\infty\} \cup (\mathbb{C} \setminus \overline{D(0, N) \cap \Omega})$ is connected.*

Proof of Lemma 5.3. Let V be a connected component of $\{\infty\} \cup (\mathbb{C} \setminus \overline{D(0, N) \cap \Omega})$. It is enough to prove that $\infty \in V$.

Now, $\{\infty\} \cup (\mathbb{C} \setminus \overline{D(0, N) \cap \Omega})$ is a non empty open set inside $\mathbb{C} \cup \{\infty\}$ and therefore, V is a non empty open set in $\mathbb{C} \cup \{\infty\}$.

That implies that there exist $x \in V$ with $|x| < N$, or there exists $x \in V$ with $|x| > N$.

In the former case, $x \in D(0, N)$ and as $x \notin \overline{(\Omega \cap D(0, N))}$, one can see that $x \notin \overline{\Omega}$. Indeed, if not there would be $(x)_{n \in \mathbb{N}} \subset \Omega$ with $x_n \rightarrow x$. But $D(0, N)$ is an open set. Thus, eventually, we have $x_n \in D(0, N)$. This implies that for n big enough we have $x_n \in (\Omega \cap D(0, N))$. It follows that $x \in \overline{(\Omega \cap D(0, N))}$, which is a contradiction.

Thus, $x \in (\mathbb{C} \setminus \overline{\Omega}) \cup \{\infty\} \subseteq \{\infty\} \cup (\mathbb{C} \setminus \overline{D(0, N) \cap \Omega})$.

Because $\{\infty\} \cup (\mathbb{C} \setminus \overline{\Omega})$ is connected, it follows that $\infty \in V$ as we wanted.

In the latter case, we have

$$x \in \{\infty\} \cup (\mathbb{C} \setminus \overline{D(0, N)}) \subseteq \{\infty\} \cup (\mathbb{C} \setminus \overline{(\Omega \cap D(0, N))}).$$

Because $\{\infty\} \cup (\mathbb{C} \setminus \overline{D(0, N)})$ is connected, it follows again this $\infty \in V$.

This completes the proof of the Lemma 5.3. ■

Lemma 5.4. *Let Ω be an open set, $\Omega \subseteq \mathbb{C}$. Then for every $N \in \mathbb{N}$ we have*

$$\overline{(\Omega \cap D(0, N))^0} = \overline{(\Omega \cap D(0, N))}.$$

Proof of the Lemma 5.4. Since $\overline{\Omega \cap D(0, N)} \supseteq \Omega \cap D(0, N)$ it follows $\overline{(\Omega \cap D(0, N))^0} \supseteq (\Omega \cap D(0, N))^0$.

This implies $\overline{(\Omega \cap D(0, N))^0} \supseteq (\Omega \cap D(0, N))$, as the latter set is open.

Therefore, $\overline{(\Omega \cap D(0, N))^0} \supseteq \overline{(\Omega \cap D(0, N))}$.

For the other inclusion let $x \in \overline{(\Omega \cap D(0, N))}^0$. Then there exist $x_n \in \overline{(\Omega \cap D(0, N))}^0$ such that $x_n \rightarrow x$. Now, for every $n \in \mathbb{N}$, there exist $\varepsilon_n \in (0, \frac{1}{n})$ such that $B(x_n, \varepsilon_n) \subseteq \overline{(\Omega \cap D(0, N))}$. As $x_n \in \overline{(\Omega \cap D(0, N))}$, for every $n \in \mathbb{N}$, there exists $y_n \in B(x_n, \varepsilon_n)$ with $y_n \in \Omega \cap D(0, N)$. But $s_n \rightarrow 0$ and $x_n \rightarrow x$ which gives $y_n \rightarrow x$. Thus, $x \in \overline{(\Omega \cap D(0, N))}$ and the proof of the Lemma 5.4 is completed. ■

Proof of the Theorem 5.2. Let $f \in A^\infty(\Omega)$, $\varepsilon > 0$, $n \in \mathbb{N}_0$ and $N \in \mathbb{N}$. It is enough to find a polynomial p , such that $\sup_{z \in \overline{(\Omega \cap D(0, N))}} |f^{(\ell)}(z) - p^{(\ell)}(z)| < \varepsilon$, for $\ell = 0, 1, \dots, n$.

Without loss of generality we can assume, $N \geq n_0$. Now let $V = \overline{((\Omega \cap D(0, N)))^0}$.

V is an open, connected and bounded set. (V is connected because $N \geq n_0$ and because of condition (c) of our hypothesis).

Also, from Lemma 5.4, we get $\overline{V} = \overline{(\overline{\Omega \cap D(0, N)})^0} = \overline{\Omega \cap D(0, N)}$ and therefore, $\overline{V^0} = \overline{(\overline{\Omega \cap D(0, N)})^0} = V$.

From Lemmas 5.3 and 5.4 it follows that $\{\infty\} \cup (\mathbb{C} \setminus \overline{V}) = \{\infty\} \cup (\mathbb{C} \setminus \overline{(\Omega \cap D(0, N))})$ is connected.

Since V is bounded, it follows that $C \setminus \overline{V}$ is connected.

Thus, V satisfies all conditions of Theorem 5.1 and therefore the set of all polynomials is dense in $A^\infty(V)$. But $f \in A^\infty(\Omega)$ and $\Omega \supset V$. Indeed we have $\Omega \supset \Omega \cap D(0, N)$. This gives $\overline{\Omega} \supset \overline{(\Omega \cap D(0, N))}$. Therefore, $\Omega = \overline{\Omega}^0 \supset (\overline{\Omega \cap D(0, N)})^0 = V$. Thus, $\Omega \supset V$.

But $\Omega \supset V$ implies $A^\infty(\Omega) \subset A^\infty(V)$.

Thus, $f \in A^\infty(V)$. Therefore, there exists a polynomial p such that

$$\sup_{z \in V} |f^{(\ell)}(z) - p^{(\ell)}(z)| < \frac{\varepsilon}{2}, \quad \text{for } \ell = 0, 1, \dots, n.$$

This implies

$$\sup_{z \in \overline{V}} |f^{(\ell)}(z) - p^{(\ell)}(z)| < \frac{\varepsilon}{2} < \varepsilon, \quad \text{for } \ell = 0, 1, \dots, n.$$

By Lemma 5.4 we have that

$$\overline{V} = \overline{((\overline{\Omega \cap D(0, N)})^0)} = \overline{\Omega \cap D(0, N)}.$$

The proof of Theorem 5.2 is complete. \blacksquare

Theorem 5.5. *Let Ω be a bounded, connected, open set such that:*

- (a) $(\overline{\Omega})^0 = \Omega$.
- (b) $\{\infty\} \cup (\mathbb{C} \setminus \overline{\Omega})$ has exactly k connected components, $k \in \mathbb{N}$.
- (c) There exists $M > 0$ such that for all $a, b \in \Omega$, there exists a continuous function $\gamma : [0, 1] \rightarrow \Omega$ with $\gamma(0) = a$, $\gamma(1) = b$ and $\text{Length}(\gamma) \leq M$.

Now pick from each connected component of $\{\infty\} \cup (\mathbb{C} \setminus \overline{\Omega})$ a point a_i , $i = 0, 1, 2, \dots, k-1$ and set $S = \{a_0, \dots, a_{k-1}\}$, where a_0 belongs to the unbounded component. Then the set of all rational functions with poles from S is dense in $A^\infty(\Omega)$ and therefore $X^\infty(\Omega) = A^\infty(\Omega)$.

For the proof we need the following lemma.

Lemma 5.6. *Let Ω be an open set, $n \in \mathbb{N}$, $n \geq 1$, and let f be holomorphic in Ω . Then, for $i \in \mathbb{N}$, $0 \leq i \leq n-1$, and γ any closed curve in Ω of bounded variation we have $\int_{\gamma} z^i f^{(n)}(z) dz = 0$.*

Proof of Lemma 5.6. We use induction on n . For $n = 1$, we have to prove $\int_{\gamma} f'(z) dz = 0$, which is obvious, because the curve γ is closed. Suppose the statement holds for $n = k$, we will prove it for $k + 1$. Let $i \in \{1, \dots, (k + 1) - 1 = k\}$, we have

$$\int_{\gamma} z^i f^{(k+1)}(z) dz = \int_{\gamma} z^i (f^{(k)}(z))' dz = z^i f^{(k)}(z) \Big|_{\gamma(0)}^{\gamma(1)} - i \int_{\gamma} z^{i-1} f^{(k)}(z) dz. \quad (1)$$

Since γ is a closed curve it follows that

$$z^i f^{(k)}(z) \Big|_{\gamma(0)}^{\gamma(1)} = 0. \quad (2)$$

Since $i - 1 < k$ from the induction hypothesis we have

$$\int_{\gamma} z^{i-1} f^{(k)}(z) dz = 0. \quad (3)$$

Relations (1), (2) and (3) imply

$$\int_{\gamma} z^i f^{(k+1)}(z) dz = 0,$$

as we wanted.

Finally, for $i = 0$ we have $\int_{\gamma} z^0 f^{(k+1)}(z) dz = \int_{\gamma} (f^{(k)}(z))' dz = 0$, because γ is closed.

The proof of the lemma is complete. \blacksquare

Proof of Theorem 5.5. We first prove it in the case $a_0 = \infty$.

Let $f \in A^\infty(\Omega)$, $\varepsilon > 0$ and $n \in \mathbb{N} = \{0, 1, 2, \dots\}$. We need to find a rational function r with poles in S such that:

$$\sup_{z \in \Omega} |f^{(\ell)}(w) - r^{(\ell)}(w)| < \varepsilon, \quad \text{for } \ell = 0, 1, \dots, n.$$

Fix curves γ_i , $i = 1, 2, \dots, k-1$ in Ω which are closed, have finite length and also $\text{Ind}(\gamma_i, a_j) = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$. For the construction of such curves see [1]. Since $f \in A^\infty(\Omega)$, it follows that $f^{(n)} \in C(\overline{\Omega})$ and that $f^{(n)}$ is analytic in $\Omega = (\overline{\Omega})^0$.

Thus, from Mergelyans' Theorem ([18], ch. 20, ex 1) there exists a rational function called $\tilde{r}_n(z)$, with poles only in S , such that $\sup_{z \in \Omega} |f^{(n)}(z) - \tilde{r}_n(z)| < a$ where $a > 0$ is sufficiently small.

In particular it suffices that

$$0 < a < \min \left(\frac{\varepsilon}{2(M+1)^n}, \frac{\varepsilon \cdot \pi \cdot \tau^n}{n(k-1) \cdot \left(D + \sum_{i=1}^{k-1} |a_i| \right)^n} \cdot \frac{1}{(M+1)^n \left(\sum_{i=1}^{k-1} \text{length}(\gamma_i) \right)} \right)$$

where $D \geq \max(1, \max_{z \in \overline{\Omega}} |z|)$ and $0 < \tau \leq \min(1, \text{dist}(S, \Omega))$.

Now, by analyzing $\tilde{r}_n(z)$ into simple fractions, there exists a rational function $r_n(z)$ with poles only in S such that

$$\tilde{r}_n(z) = r_n(z) + \sum_{i=1}^{k-1} \sum_{j=1}^n \frac{b_{ij}}{(z - a_i)^j} \quad \text{with } b_{ij} \in \mathbb{C} \quad i = 1, 2, \dots, k-1 \quad \text{and } j = 1, 2, \dots, n.$$

and

$$\text{Res}((z - a_i)^{j-1} \tilde{r}_n(z), a_i) = 0, \quad \text{for all } i = 1, 2, \dots, k-1, \quad j = 1, 2, \dots, n.$$

Fix $(i, j) \in \{1, 2, \dots, k-1\} \times \{1, 2, \dots, n\}$. Then using Lemma 5.6, it follows that

$$\begin{aligned} |b_{ij}| &= \left| \frac{1}{2\pi i} \int_{\gamma_i} (z - a_i)^{j-1} \tilde{r}_n(z) dz \right| = \left| \frac{1}{2\pi i} \int_{\gamma_i} (z - a_i)^{j-1} (\tilde{r}_n(z) - f^{(n)}(z)) dz \right| \\ &\leq \frac{1}{2\pi} (D + |a_i|)^n L(\gamma_i) \cdot a. \end{aligned}$$

Now, choosing the positive number a sufficiently small, we get

$$|b_{ij}| \leq \frac{\varepsilon \cdot \tau^n}{2n(K-1) \cdot (M+1)^n}, \quad \text{for all } i = 1, 2, \dots, K-1 \quad \text{and } j = 1, 2, \dots, n. \quad (1)$$

Since $\sup_{z \in \Omega} |f^{(n)}(z) - \tilde{r}_n(z)| < a$, it follows that

$$\sup_{z \in \Omega} \left(|f^{(n)}(z) - r_n(z)| + \sum_{i=1}^{k-1} \sum_{j=1}^n \frac{|b_{ij}|}{|z - a_i|^j} \right) < a.$$

This implies,

$$\sup_{z \in \Omega} |f^{(n)}(z) - r_n(z)| < a + \sum_{i=1}^{k-1} \sum_{j=1}^n \frac{|b_{ij}|}{|z - a_i|^j} \leq \frac{\varepsilon}{2(M+1)^n} + \sum_{i=1}^{k-1} \sum_{j=1}^n \frac{|b_{ij}|}{\tau^n}$$

Combining this with relation (1) we obtain,

$$\begin{aligned} \sup_{z \in \Omega} |f^{(n)}(z) - r_n(z)| &\leq \frac{\varepsilon}{2(M+1)^n} + \sum_{i=1}^{k-1} \sum_{j=1}^n \frac{\varepsilon \cdot \tau^{n-j}}{2n(k-1) \cdot (M+1)^n} \\ &\leq \frac{\varepsilon}{2(M+1)^n} + \frac{\varepsilon}{2(M+1)^n} = \frac{\varepsilon}{(M+1)^n}. \end{aligned}$$

The function r_n has a Laurent expansion around each $a_i \in S \setminus \{\infty\}$, where the coefficients of $(z - a)^\ell$ for $\ell = -n, -n+1, \dots, -1$ are equal to zero.

This implies that for each $s, 1 \leq s \leq n$ the integral $\int \int \cdots \int r_n(z) (dz)^s$ defines a regular holomorphic function in Ω , which is not multivalued.

We proceed by induction on $\lambda \in \{n, n-1, \dots, 0\}$. For $\lambda \in \mathbb{N}$, $0 \leq \lambda \leq n-1$, we define:

$$r_\lambda(z) = f^{(\lambda)}(z_0) + \int_{[z_0, z]} r_{\lambda+1}(z) dz,$$

where $r_{\lambda+1}$ is known by the induction hypothesis.

Thus, we define the rational functions $r_n, r_{n+1}, \dots, r_1, r_0$. We set $r = r_0$.

It is obvious that $r_\lambda(z) = r_0^{(\lambda)}(z) = r^{(\lambda)}(z)$, for $\lambda = 0, 1, \dots, n$. The proof of the case $a_0 = \infty$ can be completed as the last part of the proof of Theorem 5.1.

Next we consider the general case where a_0 is not necessarily equal to ∞ .

Let $f \in A^\infty(\Omega)$, $\varepsilon > 0$ and a natural $n \in \mathbb{N} = \{0, 1, 2, \dots\}$. We seek a rational function r with poles only in $S = \{a_0, \dots, a_{k-1}\}$, such that $\sup_{z \in \Omega} |f^{(\ell)}(z) - r^{(\ell)}(z)| < \varepsilon$ for $\ell = 0, 1, \dots, n$.

From the previous case, there exists a rational function r_1 with poles only in $\tilde{S} = \{\infty, a_1, a_2, \dots, a_{k-1}\}$ such that

$$\sup_{z \in \Omega} |f^{(\ell)}(z) - r_1^{(\ell)}(z)| < \frac{\varepsilon}{2}, \quad \text{for } \ell = 0, 1, \dots, n. \quad (2)$$

But it is known that there exists a rational function r with poles in $S = \{a_0, a_1, \dots, a_{n-1}\}$ such that

$$\sup_{z \in \Omega} |r^{(\ell)}(z) - r_1^{(\ell)}(z)| < \frac{\varepsilon}{2} \quad \text{for } \ell = 0, 1, \dots, n. \quad (3)$$

See [6], Lemma 2.2.

From relations (2) and (3) we derive that

$$\sup_{z \in \Omega} |r^{(\ell)}(z) - f^{(\ell)}(z)| < \varepsilon \quad \text{for } \ell = 0, 1, \dots, n$$

and r has its poles in S .

The proof of Theorem 5.5 is complete now. \blacksquare

The following theorem is the more general one.

Theorem 5.7. *Let Ω be a connected, open set such that:*

$$(a) \quad (\overline{\Omega})^0 = \Omega.$$

(b) $\{\infty\} \cup (\mathbb{C} \setminus \overline{\Omega})$ has exactly k connected components, $k \in \mathbb{N}$.

(c) There exists $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$, there exists $M_n > 0$ such that for all $a, b \in (\overline{(\Omega \cap D(0, n))})^0$, there exists a continuous function $\gamma : [0, 1] \rightarrow (\overline{\Omega \cap D(0, n)})^0$ with $\gamma(0) = a$, $\gamma(1) = b$ and $\text{Length}(\gamma) \leq M_n$.

Now pick from each of the k connected components of $\{\infty\} \cup (\mathbb{C} \setminus \overline{\Omega})$ a point, a_i ($i = 0, 1, \dots, k-1$) and set $S = \{a_0, a_1, \dots, a_{k-1}\}$.

Then, the set of rational functions with poles only in S is dense in $A^\infty(\Omega)$, and therefore $X^\infty(\Omega) = A^\infty(\Omega)$.

Proof. Let $r > 0$ be such that $D(0, r)$ contains all the components of $\{\infty\} \cup (\mathbb{C} \setminus \overline{\Omega})$ not containing ∞ . This is possible, since $k \in \mathbb{N}$.

Let $f \in A^\infty(\Omega)$, $\varepsilon > 0$, $n \in \mathbb{N} = \{0, 1, 2, \dots\}$ and $N \in \mathbb{N}$, $N \neq 0$. It is enough to find a rational function r with poles only in S such that:

$$\sup_{z \in (\overline{\Omega \cap D(0, N)})^0} |f^{(\ell)}(z) - r^{(\ell)}(z)| < \varepsilon,$$

because $\overline{(\overline{\Omega \cap D(0, N)})^0} = \overline{\Omega \cap D(0, N)}$ (Lemma 5.4).

Without loss of generality we may assume that $N \geq n_0 + r$. We claim that there exists $M > 0$, such that for every $a, b \in \overline{\Omega \cap D(0, N)}^0$, there exists a curve in $\overline{\Omega \cap D(0, N)}^0$ that joins a and b and has length at most M and also that the set $\{\infty\} \cup (\mathbb{C} \setminus \overline{(\Omega \cap D(0, N))})$ has at most k connected components, each of them containing at least one point from S .

The former is immediate according to our hypothesis by setting $M = M_N$.

For the latter, let V be a connected component of $\{\infty\} \cup (\mathbb{C} \setminus \overline{(\Omega \cap D(0, N))})$. Because V is open and non empty there exists $x \in V$ with $|x| > N$ or there exists $x \in V$ with $|x| < N$.

In the first case we have that $x \in \{\infty\} \cup (\mathbb{C} \setminus \overline{D(0, N)}) \subseteq \{\infty\} \cup (\mathbb{C} \setminus \overline{(\Omega \cap D(0, N))})$. Because $\{\infty\} \cup (\mathbb{C} \setminus \overline{D(0, N)})$ is connected, it follows that $\infty \in V$. Thus, the unbounded component of $\{\infty\} \cup (\mathbb{C} \setminus \overline{\Omega})$ is contained in V , which implies that $V \cap S \neq \emptyset$.

In the latter case, $x \in (\{\infty\} \cup \mathbb{C} \setminus \overline{(\Omega \cap D(0, N))}) \cap D(0, N)$. Therefore, $x \in (\mathbb{C} \setminus \overline{(\Omega \cap D(0, N))}) \cap D(0, N)$.

It follows that $x \notin \overline{\Omega}$. Indeed, if not, there exists a sequence $(x_n)_{n \in \mathbb{N}} \subset \Omega$ with $x_n \rightarrow x$. But $D(0, N)$ is open. Thus, there exists $n_0 \in \mathbb{N}$, such that $x_n \in (\Omega \cap D(0, N))$ for every $n \geq n_0$.

It follows that $x \in \overline{(\Omega \cap D(0, N))}$, contradicting the assumption that x belongs to $\mathbb{C} \setminus \overline{(\Omega \cap D(0, N))}$.

Therefore $x \in \mathbb{C} \setminus \overline{\Omega} \subseteq \{\infty\} \cup (\mathbb{C} \setminus \overline{\Omega}) \subseteq \{\infty\} \cup (\mathbb{C} \setminus \overline{(\Omega \cap D(0, N))})$.

Let V_1 be the connected component inside $\{\infty\} \cup (\mathbb{C} \setminus \overline{\Omega})$ containing x . It follows that $V_1 \subset V$ and thus, $V \cap S \neq \emptyset$.

Therefore, we have proved that any connected component of $\{\infty\} \cup (\mathbb{C} \setminus \overline{(\Omega \cap D(0, N))})$ intersects non trivially S . Since S contains exactly k points and the components of $\{\infty\} \cup (\mathbb{C} \setminus \overline{(\Omega \cap D(0, N))})$ are mutually disjoint we conclude that the number of the components is at most k . (It can also be proved that, if N is big enough, the number of components is exactly k , but this is not needed at the sequel).

Now, set $T = \overline{(\Omega \cap D(0, N))}^0$. We can easily check that T satisfies all assumptions of Theorem 5.5. It is true that $f \in A^\infty(\Omega)$. But $T \subset \overline{\Omega}^0 = \Omega$. Thus, $A^\infty(T) \supset A^\infty(\Omega)$. This implies that $f \in A^\infty(T)$. Theorem 5.5, combined with the fact that the set S contains at least one point from each component of $\{\infty\} \cup (\mathbb{C} \setminus \tilde{T})$ implies that, there exists a rational function r with poles only in S such that,

$$\sup_{z \in \tilde{T}} |f^{(\ell)}(z) - r^{(\ell)}(z)| < \varepsilon, \quad \text{for } \ell = 0, 1, 2, \dots, n.$$

It follows that,

$$\sup_{z \in (\Omega \cap D(0, N))} |f^{(\ell)}(z) - r^{(\ell)}(z)| < \varepsilon, \quad \text{for } \ell = 0, 1, 2, \dots, n,$$

because $\bar{T} = \overline{\Omega \cap D(0, N)}$, according to Lemma 5.4. The proof is complete. \blacksquare

Remark 5.8. Lemma 5.6 can be generalized to a necessary and sufficient condition for an analytic function to have an antiderivative in Ω of order n , $n \in \mathbb{N}$.

More specifically it holds the following:

Let $n \in \mathbb{N}$, Ω an open subset of \mathbb{C} and f an analytic function in Ω . The following are equivalent

- (a) There exists a function F , which is analytic in Ω , such that $F^{(n)}(z) = f(z)$, for all $z \in \Omega$.
- (b) For any closed curve γ in Ω of bounded variation, and for every $i = 0, 1, \dots, n-1$, it is true that $\int_{\gamma} z^i f(z) dz = 0$.
- (c) For any closed curve γ in Ω of bounded variation, and for every polynomial P with $\deg P \leq n-1$, it is true that $\int_{\gamma} P(z) f(z) dz = 0$.

Remark 5.9. The previous theorems remain valid for any open set $\Omega \subseteq \mathbb{C}$ with a finite number of components, where each component satisfies the prerequisites of the according theorem under the extra condition that the closures of any two components are disjoint.

The proof of this is immediate by applying our theorems in each component. This gives a finite number of rational functions, one for each component.

Applying Runge and Weierstrass theorems we find one rational function with poles off $\overline{\Omega}$ approximating simultaneously the above rational functions.

We do not have the answer in the case where the closure of the components are not disjoint but we know that the answer is positive in the particular case of two open discs D_1, D_2 such that $\overline{D_1} \cap \overline{D_2}$ is a singleton. Indeed, let $f \in A^\infty(D_1 \cup D_2)$, $\varepsilon > 0$ and $n \in \mathbb{N}$. We can assume that the disks touch at zero and also that their radius is at most 1. The open set $D_1 \cup D_2$ obviously satisfies the

- (i) $(\overline{D_1 \cup D_2})^0 = D_1 \cup D_2$
- (ii) $\mathbb{C} \setminus \overline{(D_1 \cup D_2)}$ is connected and also
- (iii) for any two points in $D_1 \cup D_2$, there exists a polygonal line joining them of length at most four, that may be touches the boundary at most at zero, and otherwise is contained in $D_1 \cup D_2$.

Now, for $f \in A^\infty(D_1 \cup D_2)$ and γ a polygonal simple curve in $\overline{D_1 \cup D_2}$, that touches the boundary at most at zero, it is immediate from an argument of continuity that

$$\int_{\gamma} f^{(\lambda)}(z) dz = f^{(\lambda)}(\gamma(1)) - f^{(\lambda)}(\gamma(0)), \quad \text{for any } \lambda \in \mathbb{N}.$$

The rest of the proof is similar to the proof of Theorem 5.1.

Remark 5.10. It can be shown that if Ω is a Jordan domain with rectifiable boundary, it fulfills the prerequisites of Theorem 5.1.

More specifically it holds that there exists a positive constant $M > 0$ such that any two points in Ω can be joined by a curve inside Ω of length at most M .

Moreover the above holds in the case of a domain bounded by k disjoint Jordan curves with rectifiable boundaries.

Indeed in the case of a Jordan domain Ω with rectifiable boundary (as in [13]) every point in Ω is joined with the boundary with a segment with length at most $\text{diam}(\Omega)$.

Next two points on the boundary of Ω can be joined by subarc of the boundary with length at most the length of the boundary. Thus $M = 2\text{diam}(\Omega) + \text{length}(\partial\Omega)$.

However, the curve is not contained in Ω . According to a Theorem of Caratheodory [12] every conformal map φ from the open unit disc D onto Ω extends to a homeomorphism from \overline{D} to $\overline{\Omega}$. Further since the boundary of Ω is rectifiable, it follows that $\varphi' \in H^1$ [12], [7]. Thus we can use the image Γ by φ of a circumference $C(0, r)$, $0 < r < 1$, where r is very close to 1 and we can replace the subarc of $\partial\Omega$ by an arc of Γ ; its length is less than or equal to $\|\varphi'\|_1$ which is equal to the length of $\partial\Omega$.

When we have k disjoint Jordan curves with rectifiable boundaries, first we join the outer boundary with another boundary using a segment of minimum length (which is minimum for all boundaries). This segment is disjoint from all other boundaries. Let E_1 , be the compact set containing the two previous boundaries and the segment. We joint E_1 with some other boundary using a segment of minimum length. We continue in this way and after a finite number of steps we obtain a (connected) curve E containing all boundaries and whose all other points belong to Ω . The length of E is finite. If we consider to points $z_1, z_2 \in \Omega$ we join each one of them with some boundary using two segments. Then we joint z_1, z_2 by these two segments and a piece of E . The length does not exceed $2\text{diam}(\Omega) + \text{length}(E)$. For the arcs contained in the outer boundary we can use the conformal mapping $\varphi : D \rightarrow \Omega'$, where Ω' is the union of Ω with all bounded component of $\mathbb{C} \setminus \overline{\Omega}$, which is simply connected.

Thus, without increasing the length we can have a curve joining z_1, z_2 in $\overline{\Omega}$ not meeting the outer boundary. For another boundary γ_j let b be a point interior to γ_j . Thus, $\text{dist}(b, \overline{\Omega}) = r > 0$. Using the inversion $w = \frac{1}{z - b}$, the complement of the interior of γ_j (with ∞ included) is transformed to a bounded simply connected domain Ω'' containing 0. Using again a conformal mapping $g : D \rightarrow \Omega''$ (and g' is again in H^1) we can replace the subarc of γ_j contained in our curve by another arc inside Ω . Its length may be increased but we can have it as close to the initial length as we wish. Thus, the assumptions of Theorem 5.5 are satisfied with $M = 2\text{diam}(\Omega) + \text{length}(E) + \delta$, for any $\delta > 0$. In particular we can have

$$M = 2\text{diam}\Omega + \text{length}(E) + 1.$$

Remark 5.11. We can have examples of Jordan domains Ω without rectifiable boundary, but satisfying the assumptions of Theorem 5.1. For instance, if we consider any

starlike Jordan domain Ω then any two points may be joined in Ω by a curve consisting of two segments and therefore its length does not exceed $2\text{diam}(\Omega)$. Certainly we can arrange that the boundary of Ω has infinite length.

Further another example is the following.

Let $\varphi : [0, 1] \rightarrow \mathbb{R}$ be continuous, $c < \min\{\varphi(t) : t \in [0, 1]\}$ and $\Omega = \{(x, y) : 0 < x < 1, c < y < \varphi(x)\}$. If φ is not of bounded variation then the length of $\partial\Omega$ is infinite but the assumptions of Theorem 5.2 are satisfied.

Remark 5.12. An alternative proof of Theorem 5.5 is by using the statement of Theorem 5.1 combined by the following Laurent decomposition ([5]).

Let Ω be a domain of finite connectivity. Let V_0, V_1, \dots, V_ℓ be the components of $(\mathbb{C} \cup \{\infty\}) \setminus \Omega$, where $\infty \in V_0$. Let $f \in A^\infty(\Omega)$, then $f = f_0 + f_1 + \dots + f_\ell$ where $f_j \in A^\infty[(V_j^c)^0]$ for $j = 0, 1, \dots, \ell$ and $\lim_{z \rightarrow \infty} f_j(z) = 0$ for $j = 1, \dots, \ell$.

Remark 5.13. If Ω is a domain satisfying the assumptions of Theorem 5.1 or Theorem 5.2 or Theorem 5.5 or Theorem 5.7 or the stronger assumptions discussed in this section, then $X^\infty(\Omega) = A^\infty(\Omega)$. Therefore in these cases the results of the Section 3 and 4 become generic in $A^\infty(\Omega)$.

6 Smoothness of the integration operator

It is known that if $D = \{z \in \mathbb{C} \mid |z| < 1\}$ and $f \in H^1$ then $F(z) = \int_0^z f(\zeta) d\zeta$ has an absolute convergent Taylor series in \overline{D} (Hardy Inequality). [7], [9] Thus, in D the antiderivative of a bounded analytic function is also a bounded analytic function.

Moreover, in D the antiderivative of a function in $A(D)$ is also contained in $A(D)$.

Now, if Ω is a Jordan domain, a theorem of Caratheodory states that every Riemann conformal mapping $\phi : D \rightarrow \Omega$ extends to a homeomorphism $\phi : \overline{D} \rightarrow \overline{\Omega}$. [12] Moreover, the boundary of Ω is rectifiable if and only if $\phi' \in H^1$ [7],[12].

Combining the statements above, we see that if Ω is a Jordan domain with rectifiable boundary then the antiderivative of any bounded analytic function defined on Ω , is also a bounded analytic function on Ω . Furthermore, the antiderivative can be extended continuously to $\overline{\Omega}$. More specifically the antiderivative of every function in $A(\Omega)$ remains again in $A(\Omega)$.

We will now examine the case where the Jordan domain Ω does not have rectifiable boundary.

Proposition 6.1. *Let Ω be a Jordan domain such that there exist a constant $M < +\infty$ with the property that any two points inside Ω can be joined with a curve inside Ω of length at most M . Let f be a bounded analytic function on Ω ; then the antiderivative of f is also a bounded analytic function on Ω .*

Proof. Fix $z_0 \in \Omega$, and for every $z \in \Omega$, choose a curve γ_z in Ω that joins z_0 and z and has length at most M . Then the antiderivative $F(z)$ is equal to $\int_{\gamma_z} f(\zeta) d\zeta$ and the result easily follows. ■

It is easy to find examples of Jordan domains Ω with non rectifiable boundary that satisfy the prerequisites of Proposition 6.1, as we discussed in Remark 5.11.

For example, consider a starlike domain with no rectifiable boundary or the case of a domain

$$\Omega = \{(x, y) \mid 0 < x < 1, c < y < \tau(x)\}, \text{ where } \tau : [0, 1] \rightarrow \mathbb{R}$$

is a continuous function with no bounded variation and $c < \min_{x \in [0, 1]} \tau(x)$. We call the last domain “Domain of type $*$ ”.

Furthermore we have

Proposition 6.2. *Let Ω be a starlike domain or a domain of type $*$. Let $f \in A(\Omega)$; then the antiderivative of f belongs also to $A(\Omega)$.*

Proof. We give the proof only in the case of a Jordan domain Ω which is starlike; the proof in the case of a domain of type $*$ is similar and is omitted.

Assume that Ω is a bounded domain which is starlike with respect to a point $z_0 \in \Omega$, say $z_0 = 0$. If $f \in A(\Omega)$, it follows that f is uniformly continuous. Thus, if $\varepsilon_1 > 0$ is given, there exists $\delta > 0$, $\delta < \varepsilon_1$, so that $|f(P) - f(Q)| < \varepsilon_1$ for all $P, Q \in \Omega$ with $|P - Q| < \delta$. One antiderivative of F is given by

$$F(z) = \int_{[0, z]} f(\zeta) d\zeta = \int_0^1 f(tz) \cdot z dt \text{ for } z \in \Omega.$$

It suffices to show that F is uniformly continuous on Ω and therefore $F \in A(\Omega)$. If $z, w \in \Omega$ are such that $|z - w| < \delta$, it follows that $|tz - tw| < \delta$ for all $t \in [0, 1]$. Therefore, $|f(tz) - f(tw)| < \varepsilon_1$ and $|z - w| < \delta$. It follows that $|f(tz)z - f(tw)w| \leq |f(tz)| \cdot |z - w| + |f(tz) - f(tw)| \cdot |w| \leq \|f\|_\infty \cdot \delta + \varepsilon_1 \text{diam}(\Omega) \leq \varepsilon_1 [\|f\|_\infty + \text{diam}(\Omega)] < \varepsilon$ provided that ε_1 has been chosen so that $0 < \varepsilon_1 < \frac{\varepsilon}{\|f\|_\infty + \text{diam}(\Omega)}$. This completes the proof. ■

After these statements, it is natural to ask whether there exists a Jordan domain Ω and a function $f \in A(\Omega)$ such that the antiderivative of f is not in $A(\Omega)$.

We provide such a counter-example finding a Jordan domain Ω and a function $f \in A(\Omega)$ such that the integral of f is not even bounded inside Ω .

Proposition 6.3. *There exist a Jordan domain Ω and a function $f \in A(\Omega)$ such that the antiderivative of f is not bounded inside Ω .*

Proof. Consider the function $g : \mathbb{C} \rightarrow \mathbb{C}$, defined by $g(z) = (z-1) \exp\left(\frac{z+1}{z-1}\right)$ for $z \neq 1$ and $g(1) = 0$. According to Proposition 2.3 there exists a Jordan domain, in the upper half plane that contains an arc of the unit circle, having 1 as one of its endpoints such that g is one to one there. Call this Jordan domain V , and set $\Omega = g(V)$.

Define $f : \Omega = g(V) \rightarrow \mathbb{C}$ by $f(w) = \frac{1}{\log(1-g^{-1}(w))} \cdot \frac{1}{\exp\left(\frac{g^{-1}(w)+1}{g^{-1}(w)-1}\right)}$. It is

easy to see that $f \in A(\Omega)$.

Now, consider points z_0, z in the unit circle and in V . If the antiderivative of f was bounded, then $\left| \int_{g(z_0)}^{g(z)} f(\zeta) d\zeta \right| \leq M$, for every z in the unit circle and in V , for some constant $M < +\infty$.

The above gives $\left| \int_{z_0}^z f(g(\zeta)) g'(\zeta) d\zeta \right| \leq M$ for every z in the unit circle close enough to 1, from the upper half plane.

Thus, setting $z_0 = e^{t_0}$ and $z = e^{it}$, with t close to 0^+ , we have

$$\left| \int_{t_0}^{\tilde{t}} f(g(e^{it})) g'(e^{it}) \cdot e^{it} dt \right| \leq M, \quad \text{for every } 0 < \tilde{t} < t_0.$$

A computation gives

$$f(g(e^{it})) g'(e^{it}) e^{it} = f(g(e^{it})) \exp\left(\frac{e^{it}+1}{e^{it}-1}\right) \frac{e^{it}-3}{e^{it}-1} = \frac{e^{it}-3}{e^{it}-1} \cdot \frac{1}{\log(1-e^{it})} \cdot e^{it}.$$

Thus it must holds that

$$\left| \int_{t_0}^{\tilde{t}} e^{it} \frac{(e^{it}-3)}{(e^{it}-1)} \cdot \frac{1}{\log(1-e^{it})} dt \right| \leq M \quad (1)$$

for every \tilde{t} , with $0 < \tilde{t} < t_0$. But if we set $h(t) = e^{it} \left(\frac{e^{it}-3}{e^{it}-1} \right) \frac{1}{\log(1-e^{it})}$, then we have that h satisfies the preresquities of the Lemma 2.4.

Indeed, the Möbius function $z \rightarrow \frac{z-3}{z-1}$, sends the unit circle to the line: $\{z \in \mathbb{C} | \operatorname{Re}(z) = 2\}$ and also satisfies the fact that $\lim_{t \rightarrow 0^+} \arg\left(\frac{e^{it}-3}{e^{it}-1}\right) = \frac{\pi}{2}$. It is also obvious

that

$$\lim_{t \rightarrow 0^+} \arg(e^{it}) = 0 \quad \text{and} \quad \lim_{t \rightarrow 0^+} \arg\left(\frac{1}{\log(1 - e^{it})}\right) = - \lim_{t \rightarrow 0^+} \arg(\log(1 - e^{it})) = \pi.$$

Thus the $\lim_{t \rightarrow 0^+} \arg(h(t))$ exists. According to Lemma 2.4, one can easily check that

the integral $\int_{0^+}^{\tilde{t}} |h(t)| dt$ has the same nature with $\int_{0^+}^{\tilde{t}} \frac{1}{t|\ln t|} dt = +\infty$.

This means that $\left| \int_{t_0}^{\tilde{t}} h(t) dt \right|$ cannot be bounded for all $\tilde{t} : 0 < \tilde{t} < t_0$, yielding the desired contradiction with relation 1. The proof of Proposition 6.3 is now complete. ■

This conversation leads us to Volterra operators on $D = \{z \in \mathbb{C} \mid |z| < 1\}$. Let g be an analytic function on D . Then the operator T_g maps an analytic function f on D to the antiderivative of fg' vanishing at 0.

Open problems in this area are to characterize the functions g such that for all $f \in H^\infty(D)$, it holds that $T_g(f) \in H^\infty(D)$, and also to characterize the functions g , such that for all $f \in A(D)$, it holds $T_g(f) \in A(D)$ see [2]. It is obvious that if $g' \in H^1$ then both are satisfied. If Ω is a starlike domain or a domain of type $*$ without rectifiable boundary, then Proposition 6.2 yields for the Riemann mapping $g : D \rightarrow \Omega$, that $T_g(H^\infty(D)) \subset H^\infty(D)$ despite the fact that $g' \notin H^1$.

Moreover, it also holds $T_g(A(D)) \subset A(D)$.

In a more general way, if for a Jordan domain it holds that there exists $M < +\infty$, such that every two points in Ω can be joined by a curve inside Ω with length at most M , then the Riemann mapping $g : D \rightarrow \Omega$ satisfies $T_g(H^\infty(D)) \subset H^\infty(D)$, according to Proposition 6.1. This happens, more specifically, even if the boundary of Ω is not rectifiable.

7 Some generic results

In the case of the Jordan domain Ω constructed in Proposition 6.3 the set of functions $f \in A(\Omega)$, such that their antiderivative F is not bounded is not void and in fact it is G_δ and dense in $A(\Omega)$. This follows from the following proposition.

Proposition 7.1. *Let Ω be a Jordan domain in \mathbb{C} . We consider the sets.*

$X_1(\Omega) = \{f \in A(\Omega) : \text{The antiderivative of } f \text{ does not belong to } H^\infty(\Omega)\}.$

$X_2(\Omega) = \{f \in A(\Omega) : \text{The antiderivative of } f \text{ does not belong to } A(\Omega)\}.$

$X_3(\Omega) = \{f \in H^\infty(\Omega) : \text{The antiderivative of } f \text{ does not belong to } H^\infty(\Omega)\} \text{ and}$

$X_4(\Omega) = \{f \in H^\infty(\Omega) : \text{The antiderivative of } f \text{ does not belong to } A(\Omega)\}.$

Then we have

i) If $X_1(\omega) \neq \emptyset$, then $X_1(\Omega)$ is dense and G_δ in $A(\Omega)$ and $X_2(\Omega)$ is residual in $A(\Omega)$ and

ii) If $X_3(\Omega) \neq \emptyset$, then $X_3(\Omega)$ is dense and G_δ in $H^\infty(\Omega)$ and $X_4(\Omega)$ is residual in $H^\infty(\Omega)$.

Proposition 7.1 follows easily from the following.

Proposition 7.2. For $g \in H(D)$ we consider the Volterra operator $T_g : H(D) \rightarrow H(D)$, where $T_g(f)$ is the antiderivative of fg' vanishing at 0 for any $f \in H(D)$. We consider the following sets:

$$Y_1(g) = \{f \in A(D) : T_g(f) \notin H^\infty(D)\}$$

$$Y_2(g) = \{f \in A(D) : T_g(f) \notin A(D)\}$$

$$Y_3(g) = \{f \in H^\infty(D) : T_g(f) \notin H^\infty(D)\} \text{ and}$$

$$Y_4(g) = \{f \in H^\infty(D) : T_g(f) \notin A(D)\}.$$

Then we have

i) If $Y_1(g) \neq \emptyset$, then $Y_1(g)$ is dense and G_δ in $A(D)$ and $Y_2(g)$ is residual in $A(D)$.

ii) If $Y_3(g) \neq \emptyset$, then $Y_3(g)$ is dense and G_δ in $H^\infty(D)$ and $Y_4(g)$ is residual in $H^\infty(D)$.

Proof. We have the following description of $Y_1(g) : Y_1(g) = \bigcap_{M \in \mathbb{N}} E_M(g)$, where $E_M(g) = \{f \in A(D) : \|T_g(f)\|_\infty > M\}.$

First we will show that $E_M(g)$ is open in $A(\Omega)$; this will imply that $Y_1(g)$ is G_δ in $A(\Omega)$. Let $f \in E_M(g)$; then there exists $z_0; |z_0| < 1$, such that $\left| \int_0^{z_0} f(\zeta)g'(\zeta)d\zeta \right| > M.$

Let $\varepsilon_1 > 0$ to be defined later. If $\tilde{f} \in A(\Omega)$ satisfies $\|f - \tilde{f}\|_\infty < \varepsilon_1$ then $|T_g(\tilde{f})(z_0) - T_g(f)(z_0)| \leq \varepsilon_1 \sup_{|\zeta| \leq |z_0|} |g'(\zeta)| \cdot |z_0| \leq \varepsilon_1 \sup_{|\zeta| \leq |z_0|} |g'(\zeta)|.$ Thus, $|T_g(\tilde{f})(z_0)| \geq |T_g(f)(z_0)| - \varepsilon_1 \sup_{|\zeta| \leq |z_0|} |g'(\zeta)|.$

We choose $\varepsilon_1 > 0$, such that, $|T_g(f)(z_0)| - \varepsilon_1 \sup_{|\zeta| \leq |z_0|} |g'(\zeta)| > M$. It follows $\|T_g(\tilde{f})\|_\infty > M$ and $\tilde{f} \in E_M(g)$. Therefore $E_M(g)$ is open in $A(\Omega)$.

Next we show that $E_M(g)$ is dense in $A(\Omega)$. If we do so then Baire's Category Theorem will complete the proof. Let $w \in A(D)$ and $\varepsilon > 0$; we are looking for a function $f \in A(\Omega)$ such that $\|w - f\|_\infty < \varepsilon$ and $\|T_g(f)\|_\infty > M$.

Since $Y_1(g) \neq \emptyset$, there exists a function $\ell \in A(\Omega)$ such that $\|T_g(\ell)\|_\infty = +\infty$. It suffices to set $f = w + \varepsilon_1 \ell$ where $\varepsilon_1 > 0$ is sufficiently small. Then $\|T_g(f)\| = +\infty > M$.

It follows that $E_M(g)$ is G_δ dense in $A(\Omega)$. Since $Y_1(g) \subseteq Y_2(g)$ the proof of i) is complete.

The proof of ii) is similar and is omitted. ■

Next we have the following.

Proposition 7.3. *Let $X = H(D)$ endowed with the topology of uniform convergence on each compact subset of D . Or $X = A(D)$ endowed with the supremum norm.*

Then the sets $\{g \in X : Y_1(g) \neq \emptyset\} \equiv L_1(X)$ and $\{g \in X : Y_3(g) \neq \emptyset\} \equiv L_2(X)$ are dense in X .

Proof. We consider the Jordan domain Ω of Proposition 6.3 and let $g_0 : D \rightarrow \Omega$ be a Riemann map of D onto Ω . Then $g_0 \in A(D) \subset X$ and $Y_1(g_0) \neq \emptyset$. Let $f_0 \in A(D) : T_{g_0}(f_0) \notin H^\infty(D)$. We also have $Y_3(g_0) \neq \emptyset$.

We will show that $L_1(X)$ is dense in X . Let $\omega \in X$. If $Y_1(\omega) \neq \emptyset$ then $\omega \in L_1(X) \subset \overline{L_1(X)}$. Suppose $Y_1(\omega) = \emptyset$. Then $T_\omega(f_0) \in H^\infty(D)$. It follows that $T_{\omega + \varepsilon g_0}(f_0) = T_\omega(f_0) + \varepsilon T_{g_0}(f_0) \notin H^\infty(D)$ for all $\varepsilon > 0$. Since $\lim_{\varepsilon \rightarrow 0} \omega + \varepsilon g_0 = \omega$ and $\omega + \varepsilon g_0 \in L_1(X)$, it follows $\omega \in \overline{L_1(X)}$. Thus, $L_1(X)$ is dense in X . The proof that $L_2(X)$ is dense in X is similar. ■

In Proposition 7.3 we wonder if $L_1(X)$ and $L_2(X)$ are also G_δ in X . We also wonder if we can find a complete metric topology in the set of all Jordan domains (contained in a closed disc), so that generally for all such Jordan domains Ω the result of Proposition 6.3 holds.

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